THE EXPONENTIAL AND HYPERBOLIC FUNCTIONS

AND THEIR APPLICATIONS

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AND THEIR APPLICATIONS

A PRACTICAL BOOK FOR THE
GENERAL STUDENT AND ENGINEER

BY

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"Mathematics and Music, two adorable sisters, the harmonious daughters of Number and Imagination."

This little book is inscribed to a small group of Junior Technical School pupils who willingly submitted to this experiment in Mathematics.

PREFACE

In this book the treatment of those useful and interesting functions known as the exponential and hyperbolic functions is not intended to be either academic or exhaustive, but is designed specially to meet the requirements of students more concerned with the applications than with the intricate niceties of the theory of these functions.

The object is to guide the reader quickly and easily from elementary mathematics to these important relations without burdening the mind with pedantic proofs and remote possibilities.

Only an elementary knowledge of trigonometry, differentiation, and integration is assumed. The relations established are of great use to students of applied science, especially of electrical engineering, physics, chemistry, and even economics, but students of mathematics also will find the sequence interesting.

With this knowledge of mathematics the student will be able to read without difficulty such modern works as *Electrical Power Transmission and Interconnection* (Dannatt and Dalgleish, Pitman), and the many American publications on advanced electrical engineering.

The sections marked with an asterisk may be omitted without breaking the continuity of the subject.

CONTENTS

ABBREVIATIONS AND SYMBOLS

log : Common logarithms (base 10)

logh : Napierian or hyperbolic logarithms (base e)

j : Square root of minus one

$\simeq$: Nearly equals

! : Factorial; e.g. $4! = 4 \times 3 \times 2 \times 1$

$\therefore$: Therefore

i.e. : That is

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CHAPTER I

THE EXPONENTIAL FUNCTION

§1. THE primary object is to find a series in powers of x which on differentiation gives the same series.

The series obviously may be of the form

$$x^\circ, \quad x^1, \quad x^2, \quad x^3, \quad x^4, \quad x^5, \text{ etc.} \quad . \quad . \quad (1)$$

The differential coefficients of these terms are, respectively:

$$0, \quad 1, \quad 2x, \quad 3x^2, \quad 4x^3, \quad 5x^4, \text{ etc.} \quad . \quad . \quad (2)$$

It only remains to decide what numerical coefficients will make the two series alike.

Write a, b, c, etc., as the coefficients of x, x^2, x^3, etc., respectively; then since $x^\circ = 1$, series (1) becomes

$$1, \quad ax, \quad bx^2, \quad cx^3, \quad dx^4, \quad ex^5, \text{ etc.} \quad . \quad . \quad (3)$$

and by differentiation we obtain the series

$$a, \; 2bx, \; 3cx^2, \; 4dx^3, \; 5ex^4, \text{ etc.} \quad . \quad . \quad . \quad (4)$$

For the last two series (3 and 4) to be alike, it is seen by comparison that

$$\begin{array}{l|l|l|l}
a = 1 \quad 2b = a = 1 & 3c = b = \dfrac{1}{2} & 4d = c = \dfrac{1}{2 \, . \, 3} & \\
\qquad\qquad \therefore\; b = \dfrac{1}{2} & \therefore \quad c = \dfrac{1}{2 \, . \, 3} & \therefore \quad d = \dfrac{1}{2 \, . \, 3 \, . \, 4} & \text{etc.} \\
 & \qquad = \dfrac{1}{3\,!} & \qquad = \dfrac{1}{4\,!} &
\end{array}$$

Hence the series required is:

$$1, \ x, \ \frac{x^2}{2!}, \quad \frac{x^3}{3!}, \quad \frac{x^4}{4!}, \quad \frac{x^5}{5!} \text{ etc.} \tag{5}$$

This series is known as the **Exponential Series.**

§2. **Properties.** The differential coefficient of any term is equal to the term immediately in front of that term. For example:

$$\frac{d}{dx}\left(\frac{x^4}{4!}\right) = \frac{x^3}{3!}$$

Now the nth term of the series is $\dfrac{x^{n-1}}{(n-1)!}$ and the $(n+1)$th term is $\dfrac{x^n}{n!}$.

The ratio of the $(n+1)$th term to the nth term is, by division, $\dfrac{x}{n}$, which may be made as small as we please by making n correspondingly large.

In other words, series (5) is convergent and its sum has a limiting value.

The fact that the series obtained by differentiating (5) *has a term less than series* (5) *is of no consequence when we are dealing with the sum of the whole series.*

§3. **The Sum of the Exponential Series.** Let the sum of the series when $x = 1$ be denoted by the symbol e, then

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \text{etc., to infinity.} \tag{6}$$

The value of this can be worked out to any desired place of decimals. To five places,

$$e = 2{\cdot}71828.$$

It will now be shown that the sum of the series in x, i.e. series (5), is equal to e^x.

Consider the following:

If we multiply a few terms of the series in x by a few corresponding terms of the same series, but in y, and

arrange the products according to degree, we obtain a series in $(x + y)$ thus:

$$\left(1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \text{etc.}\right) \times \left(1 + y + \frac{y^2}{2\,!} + \frac{y^3}{3\,!} + \text{etc.}\right)$$

$$= 1 + x + y + \frac{x^2}{2\,!} + xy + \frac{y^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^2y}{2\,!} + \frac{xy^2}{2\,!} + \frac{y^3}{3\,!} + \text{etc.}$$

$$= 1 + (x + y) + \frac{(x + y)^2}{2\,!} + \frac{(x + y)^3}{3\,!} + \text{etc.} \qquad (7)$$

If now we take x equal to 1 and also y equal to 1 on both sides of this relation (7) we obtain the result $e \times e$, i.e. of e^2, namely,

$$e^2 = 1 + 2 + \frac{2^2}{2\,!} + \frac{2^3}{3\,!} + \text{etc.}$$

and, similarly, if x is 2 and y is 1 we get

$$e^3 = 1 + 3 + \frac{3^2}{2\,!} + \frac{3^3}{3\,!} + \text{etc.}$$

and, generally, we may write for any exponent x

$$\boldsymbol{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}} + \text{etc.} \qquad . \qquad . \qquad (8)$$

and for the exponent y

$$e^y = 1 + y + \frac{y^2}{2\,!} + \frac{y^3}{3\,!} + \text{etc.}$$

Result (7) is now seen to be consistent with the statement

$$e^x \times e^y = e^{(x+y)}$$

in which e^x, e^y, and $e^{(x+y)}$ represent the series given, and e is the sum of series (6), i.e. approximately 2·71828.

Further, if we write $\boldsymbol{kx}$ for $\boldsymbol{x}$ in (8) $\boldsymbol{k}$ being a constant, we obtain the series

$$\boldsymbol{e^{kx} = 1 + kx + \frac{k^2x^2}{2!} + \frac{k^3x^3}{3!}} + \text{etc.}$$

§4. ANY NUMBER CAN BE EXPRESSED IN THE EXPONENTIAL FORM.

Take 6 for example.

Let $e^x = 6$, then $x = \dfrac{\log 6}{\log e} = \dfrac{\cdot 7782}{\cdot 4343} = 1\cdot 79$ approx.

$$\therefore \quad 6 = e^{1\cdot 79}$$

$$= 1 + 1\cdot 79 + \frac{(1\cdot 79)^2}{2\,!} + \frac{(1\cdot 79)^3}{3\,!} + \frac{(1\cdot 79)^4}{4\,!} \text{ etc.}$$

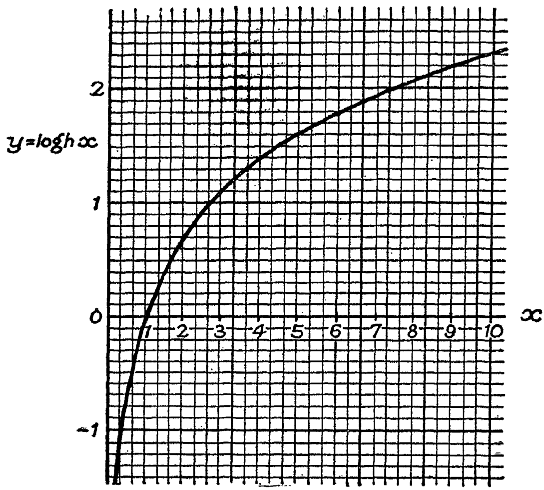

FIG. 1

§5. **Logarithms to the Base e.** Referring to the above numerical example, it is seen that 1·79 is the power to which e must be raised to give 6. In other words, 1·79 is the logarithm of 6 to the base e, or briefly,

$$1\cdot 79 = \text{logh } 6.$$

Logarithms to this base are more important, mathematically, than those to the base 10.

They are called the NATURAL or NAPIERIAN logarithms, and were invented by John Napier about the year 1614.

Fig. (1) shows the graph of logh x.

§6. **Napierian and Common Logarithms.** Napierian logarithms = common logs × 2·3026.

Proof—

$$\text{Let} \qquad z = \text{logh } a,$$

$$\text{then } e^z = a$$

and taking logs to base 10,

$$z \log e = \log a$$

$$\therefore \qquad z = \frac{\log a}{\log e} = \frac{\log a}{\cdot 4343}$$

$$= \log a \times 2{\cdot}3026,$$

$$\text{i.e. } \mathbf{logh}\ \boldsymbol{a} = \mathbf{log}\ \boldsymbol{a} \times \mathbf{2{\cdot}3026}$$

For the fundamental calculation of Napierian logarithms see page 40.

It is sometimes convenient to write a term such as "a" in the form $e^{\text{logh } a}$, or the term a^x as $e^{x \text{ logh } a}$.

§7. **Series form of $\boldsymbol{e^{-x}}$**

$$\text{That } \boldsymbol{e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}} - \text{etc.} \qquad (9)$$

i.e. the series obtained by substituting $-x$ for x in the series of e^x, may be verified by dividing out to a few terms the reciprocal of e^x, namely,

$$\frac{1}{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{etc.}}$$

$$\left(\text{Remember that } e^{-x} = \frac{1}{e^x}\right)$$

The graphs of e^x and e^{-x} are shown in Fig. 4.

Observe that both e^x and e^{-x} are positive for all values of x.

EXERCISES

1. Arrange each of the following as an exponential series: (i) 2, (ii) 10, (iii) 0·3, (iv) $\sqrt{2}$.

2. From the fact that $\log(e^3) = 3\log e$, calculate e^3. Similarly find $e^{\frac{1}{3}}$, $e^{\sqrt{2}}$, $\sqrt{e}$, $e^{\frac{\pi}{2}}$, $e^{\frac{-\pi}{2}}$.

3. Find z such that $e^z = \sin 30°$.

4. Show that $1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \text{etc.} = 2 - e^x$.

5. Write e^{-1} in series form, and from this series find e^{-1} to 4 places of decimals. Check your result by comparing it with the quotient $\frac{1}{2 \cdot 71828}$.

6. Write down series (5) omitting the first term, and underneath write the geometrical series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \text{etc.}$$

Compare the two series term by term and verify that the first is less than the second. Hence show that the sum of the first is less than 2 and that e is therefore less than 3.

Fig. 2 shows how the sums of the terms of these two series increase with the number of terms added together. For the same number

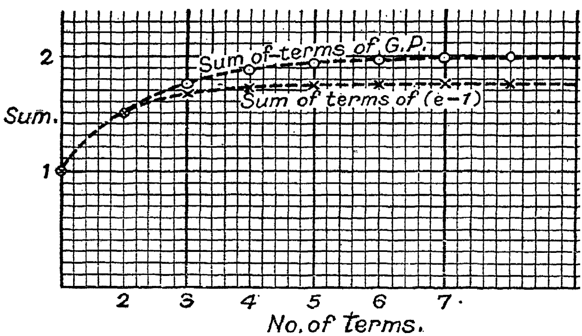

FIG. 2

Graphical comparison of the series

$$(e-1) = 1 + \frac{1}{2!} + \frac{1}{3!} + \text{etc.}$$

and the G.P. $\quad 1 + \frac{1}{2} + \frac{1}{4} + \text{etc.}$

of terms the first is less than the second excepting for the first two terms when they are equal.

§8. **Differentiation and Integration of Exponentials.** From fundamental considerations, series (8) we may write

(i) $$\frac{d}{dx}(e^x) = e^x \tag{10}$$

and $$\int e^x dx = e^x + C \tag{11}$$

(ii) If $y = e^{kx}$, then $\frac{dy}{dx} = ke^{kx} = ky$, ,

i.e. $$\frac{d}{dx}(e^{kx}) = ke^{kx} = ky \tag{12}$$

This can be shown by differentiating the series

$$1 + kx + \frac{k^2x^2}{2!} + \frac{k^3x^3}{3!} \text{ etc.}$$

thereby obtaining

$$k\left(1 + kx + \frac{k^2x^2}{2!} + \frac{k^3x^3}{3!} \text{ etc.}\right) = ke^{kx} = ky;$$

or by putting $kx = z$ and applying the well-known relation

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

From (12) it follows that

$$\int e^{kx} dx = \frac{e^{kx}}{k} + C \tag{13}$$

(iii) If $y = \operatorname{logh} x$, then $x = e^y$

and $$\frac{dx}{dy} = e^y = x$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x}$$

i.e. $$\frac{d}{dx}(\text{logh } x) = \frac{1}{x} \quad . \tag{14}$$

and ∴ $$\int \frac{dx}{x} = \text{logh } x + C \tag{15}$$

The following are also easily proved:

(iv) $\frac{d}{dx}(e^{-kx}) = -ke^{-kx}$

$$\int e^{-kx}\, dx = -\frac{e^{-kx}}{k} + C \quad . \tag{16}$$

(v) $\frac{d}{dx}(e^{k-ax}) = -ae^{k-ax}$

$$\int e^{k-ax}\, dx = -\frac{e^{k-ax}}{a} + C \tag{17}$$

(vi) $\frac{d}{dx}(\text{logh}(k \pm x)) = \pm\frac{1}{k \pm x}$

$$\int \frac{dx}{k \pm x} = \pm\, \text{logh}(k \pm x) + C \tag{18}$$

(vii) $\frac{d}{dx}(a^x) = \frac{d(e^{x \text{ logh } a})}{dx}$

$= (\text{logh } a)e^{x \text{ logh } a}$

$= (\text{logh } a)a^x$

$$\int a^x dx = \frac{a^x}{\text{logh } a} + C \quad . \tag{19}$$

EXERCISES

1. Show that $\frac{d}{dx}(e^{-x^{-2}}) = 2x^{-3}\, e^{-x^{-2}}$.

2. Show that $\frac{d}{dx}(\text{logh}(K \pm ax)) = \pm\frac{a}{K \pm ax}$
and that
$$\int \frac{dx}{K \pm ax} = \pm\frac{1}{a}\,\text{logh}(K \pm ax) + C.$$

3. Find the area between the graph e^x and the axis of x from $x = 0$ to $x = 1$.

4. Find the area under the graph of $xy = k$ or $y = \frac{k}{x}$ from $x = 1$ to $x = 3$ and generally from $x = x_1$ to $x = x_2$.

5. The difference in potential (V) between two coaxial cylinders of radii a and b when Q is the charge per unit length of one of them and the other is earthed, is:

$$V = \int_a^b \frac{2Q}{r} \, . \, dr$$

The capacity per unit length is $\frac{Q}{V}$. Show that $\frac{Q}{V}$ equals $\dfrac{1}{2 \operatorname{logh} \frac{b}{a}}$.

6. The potential at a point distant r_1 and r_2 from two thin parallel conductors carrying charges $+q$ and $-q$ respectively, per unit length is $\int_{r_1}^{s} \frac{2q}{Kr} \, . \, dr - \int_{r_2}^{s} \frac{2q}{Kr} \, dr$. Show that this equals $\frac{2q}{K} \operatorname{logh} \frac{r_2}{r_1}$.

7. The work done when a gas expands from volume v_1 to volume v_2 is $\int_{v_1}^{v_2} P dv$, in which P represents pressure. If $Pv = K$ (constant), show that the work is $K \operatorname{logh} \frac{v_2}{v_1}$.

8. Show that $\frac{d}{dx}(x \operatorname{logh} x) = \operatorname{logh} x + 1$, and hence verify that $\int \operatorname{logh} x \, dx = x \operatorname{logh} x - x$.

9. Show that $\dfrac{a}{x(a-x)} = \dfrac{1}{x} + \dfrac{1}{a-x}$

and that $\displaystyle\int \frac{dx}{x(a-x)} = \frac{1}{a} \operatorname{logh} \frac{x}{a-x} + C$

CHAPTER II

APPLICATIONS

§1. RESULT (12) Chapter I, namely: if $y = e^{kx}$ then $\dfrac{dy}{dx} = ky$ is of interest in that it illustrates a function the rate of change of which is proportional to the function.

The function e^{kx} has values ranging from the infinitesimal (when x is negative and numerically large), through finite values to the infinite. (See the graph of e^x.)

It may be taken to represent growth in which the rate of growth at any value is proportional to that value. Such growths are met with occasionally. If in the case of a sum of money put out to interest the interest were added every moment, we should have an example, since the rate at which the amount would grow would be proportional to the sum standing to the investor's credit at that moment.

If k is the rate of increase, say per £ per annum; t the time in years; P the principal, and A the amount, then

$$A = Pe^{kt}, \text{ since } \frac{dA}{dt} = kPe^{kt} = kA$$

i.e. the rate of growth is proportional to A.

Consider the following examples.

EXAMPLE 1

To find the sum to which £1 will amount in 10 years at the rate equivalent to £·05 per £ per annum, (i.e. 5% per annum) the interest being added every moment.

Let £A be the amount,

then $\quad A = 1 \times e^{\cdot 05 \times 10} = e^{\cdot 5}$

$$\text{Log } A = \cdot 5 \log e = \cdot 5 \times \cdot 4343 = \cdot 2172$$

$\therefore \quad A = £1{\cdot}649.$

When the interest is added yearly the result is

$$A = 1 \times (1{\cdot}05)^{10} = £1{\cdot}629.$$

Verify that by 20 years' continuous growth £1 would become $£e = £2{\cdot}71828$, whereas if interest were added yearly £1 would become £2·655, while at simple interest it would be merely doubled.

EXAMPLE 2

At a particular instant there is in the bank to a person's credit say £1,000. Two years later the amount is £1,100. If interest were continually accruing find (*a*) the rate of growth, (*b*) how long it has been accumulating if the original sum were £1, and (*c*) the approximate increase in the subsequent week.

Note. It is sometimes necessary to remind students that interest is increase, and rate of interest = rate of increase.

Solution—

$$e^{kt} = £1{,}000$$
$$e^{k(t+2)} = £1{,}100.$$

where k is the rate of interest and t is the time to accumulate £1,000.

By division $\quad e^{2k} = 1{\cdot}1$

and by logs $\quad 2k = \text{logh } 1{\cdot}1$

$$k = \frac{\text{logh } 1{\cdot}1}{2} = \frac{\cdot 0414 \times 2{\cdot}3026}{2}$$

$$= \cdot 04766 \text{ or } 4{\cdot}766\%. \qquad (a)$$

From the first equation

$$t = \frac{\text{logh } 1000}{k} = 144{\cdot}9 \text{ years} \qquad (b)$$

Let δA be the increment in one week, then since

$$A = e^{kt} \text{ and } \frac{dA}{dt} = ke^{kt} = kA$$

and
$$\delta A \simeq \frac{dA}{dt} \,.\, \delta t$$

$$\delta A \simeq \cdot 04766 \times 1100 \times \frac{1}{52} = £1{\cdot}008 \qquad (c)$$

Examples 1 and 2 are rather hypothetical in that banks do not add interest in this manner, but they are chosen to illustrate continuous growth and what can be calculated from given data.

EXAMPLE 3

The natural rate of increase in population at any time is considered to be proportional to the population at that time:

That is, if P is the population

$$\frac{dP}{dt} = kP$$

$$\int \frac{dP}{P} = \int k dt$$

i.e. $$\operatorname{logh} P = kt + C.$$

If P_o is the population at time $t = o$ then $C = \operatorname{logh} P_o$, and $\operatorname{logh} P - \operatorname{logh} P_o = kt$

or $$\operatorname{logh} \frac{P}{P_o} = kt,$$

or $$P = P_o e^{kt}.$$

Numerical Example. The population of England and Wales in 1881 was 25·974 millions and in 1891, 29·003 millions. What should it have been in 1901 if the increase had followed the natural law of increase stated above?

Solution—

Let P_o = population in 1881, in millions,

P_1 = population in 1891, in millions,

P = estimated population in 1901, in millions.

First determine k from the relation

$P_1 = P_o e^{kt}$ in which $t = 10$.

By common logs

$$k = \frac{\log P_1 - \log P_o}{10 \log e}$$

$$= \frac{1{\cdot}4624 - 1{\cdot}4145}{4{\cdot}343}$$

$$= {\cdot}01103.$$

Then calculate P from the relation $P = P_o e^{\cdot 01103 \times 20}$.

Again by common logs:

$$\log P = \log P_o + {\cdot}2206 \log e$$
$$= 1{\cdot}4145 + {\cdot}0958,$$
$$= 1{\cdot}5103.$$

By antilogs, $P = 32{\cdot}38$ millions.

The actual census population was 32·528 millions.

As an exercise, calculate the population for 1911.

EXAMPLE 4

In economics the problem of depreciation arises in such cases as the value of property and machinery. There are instances in which the rate of depreciation is proportional to the value at the time of consideration. Symbolically, if v is the value at the time t then

$$\frac{dv}{dt} = -kv \text{ (minus because decreasing)}$$

and
$$\int \frac{dv}{v} = -k \int dt$$

i.e.
$$\text{logh } v = -kt + C.$$

If V is the initial value, i.e. at $t = 0$

then
$$C = \text{logh } V$$

and $\therefore$
$$\text{logh } v = -kt + \text{logh } V$$

or
$$\text{logh } \frac{v}{V} = -kt$$

or
$$v = Ve^{-kt}.$$

Numerical Example. Suppose that a machine costing £500 is worth only scrap price, say £5, in 10 years; find the value at the end of 5 years.

We have, $V = 500$, $t = 10$, and $v = 5$.

$$\therefore \qquad k = -\operatorname{logh} \frac{5}{500} \Big/ 10$$

$$= \frac{\operatorname{logh} 100}{10} = \cdot 4605.$$

For $t = 5$,

$$v = 500\, e^{-\cdot 4605 \times 5}$$

$$= 500\, e^{-2 \cdot 3025}$$

$$= 500 \times \frac{1}{10}$$

$$= 50,$$

i.e. in 5 years the value is £50.

As an exercise, find (i) the value of the above machine after 2 years, (ii) the time in which it depreciates to half its original value.

EXAMPLE 5

A body moves so that its velocity in ft. p. sec. is numerically equal to one-tenth its distance in ft. from a fixed point. Find expressions for its position and velocity at any instant.

If $s =$ the distance (ft.) from the fixed point at the time t (sec.)

then
$$v = \frac{ds}{dt} = \frac{s}{10}$$

$$\frac{ds}{s} = \frac{1}{10} dt,$$

and by integration

$$\operatorname{logh} s = \frac{t}{10} + C.$$

To find C it is useless to put $s = 0$ for $\text{logh } 0 = -\infty$. If, however, we represent the distance of the body from the fixed point when $t = 0$ by S_o, then,

$$C = \text{logh } S_o$$

and
$$\text{logh } s = \frac{t}{10} + \text{logh } S_o$$

or
$$\text{logh } \frac{s}{S_o} = \frac{t}{10}$$

or
$$s = S_o e^{\frac{t}{10}}$$

which gives its position.

Further, $$v = \frac{ds}{dt} = \frac{S_o}{10} e^{\frac{t}{10}}$$

which gives its velocity. This is seen to equal

$$\frac{s}{10}$$

For example, if the body is 1 ft. from the fixed point when $t = 0$, we can find the time and velocity when s is, say 20, and check the relation; thus from $s = S_o e^{\frac{t}{10}}$

$$\begin{aligned} t &= 10 \text{ logh } 20 \\ &= 10 \times 1{\cdot}301 \times 2{\cdot}3026 \\ &= 29{\cdot}957 \text{ sec.} \end{aligned}$$

$$v = \frac{1}{10} e^{2 \cdot 9957}$$

$$\begin{aligned} \therefore \quad \log v &= \log {\cdot}1 + 2{\cdot}9957 \log e \\ &= \bar{1} + 2{\cdot}9957 \times {\cdot}4343 \\ &= \bar{1} + 1{\cdot}3010 \\ &= {\cdot}3010 \end{aligned}$$

$$\therefore \quad v = 2 \text{ which is a tenth of 20.}$$

EXAMPLE 6

By Newton's law of cooling the rate of fall in temperature of a body is proportional to the excess of its temperature over that of its surroundings.

If θ = excess temperature and t denotes time

then $$\frac{d\theta}{dt} = -k\theta$$

and, therefore, by integration

$$\operatorname{logh}\theta = -kt + C$$

If θ_o is the excess temperature when $t = 0$

then $$C = \operatorname{logh}\theta_o$$

and $$\theta = \theta_o e^{-kt}$$

Exercise: The temperature of a cooling body is observed to be 100° C. and two minutes later to be 80° C. If the temperature of the surroundings is kept constant at 20° C., find the temperature of the body at the end of another three minutes.

EXAMPLE 7

The velocity of chemical reaction is proportional to the concentration of the substance. (Guldberg and Waage's Law.)

If s = original amount of substance per unit volume

and x = amount transformed in time t

then $s - x$ = amount remaining at end of time t

and $$\frac{dx}{dt} = k(s - x)$$

$$\therefore \quad \frac{dx}{s - x} = k\,dt$$

and by integration

$$-\operatorname{logh}(s - x) = kt + C,$$

Since x is o when t is o, $C = -\text{logh}\, s$

$$-kt = \text{logh}\,\frac{s-x}{s}$$

or
$$\frac{s-x}{s} = e^{-kt}$$

from which,
$$x = s(1 - e^{-kt})$$

From known simultaneous values of x, s and t, k can be determined and used to find other values of x.

EXAMPLE 8

The following is met in the study of electric currents.

(i) On applying a constant e.m.f. (E) to a circuit of resistance (R) and self-induction (L), the current (i) grows at the rate given in the equation:

$$L\frac{di}{dt} + Ri = E$$

and from this the value of i at any time t can be determined.

Transposing,
$$\frac{di}{dt} = \frac{R}{L}\left(\frac{E}{R} - i\right)$$

$$\frac{di}{\frac{E}{R} - i)} = \frac{R}{L}dt$$

By integration

$$-\text{logh}\left(\frac{E}{R} - i\right) = +\frac{R}{L}t + C$$

Now when t is 0, i is 0, $\therefore\ C = -\text{logh}\,\frac{E}{R}$

$$\therefore \qquad \text{logh}\left(\frac{E}{R} - i\right) - \text{logh}\,\frac{E}{R} = -\frac{R}{L}t$$

or $$e^{-\frac{Rt}{L}} = \frac{\frac{E}{R} - i}{\frac{E}{R}}$$

or $$i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right).$$

It is explained in books on electricity that $\frac{E}{R}$ is the full or ultimate value reached by the current.

(ii) When the circuit is broken E is zero and the current falls in value. The given equation reduces to,

$$\frac{di}{dt} = -\frac{R}{L}i$$

or $$\frac{di}{i} = -\frac{R}{L}dt$$

and by integration

$$\text{logh}\, i = -\frac{R}{L}t + c$$

If I is the value of the current when $t = 0$, i.e. just the instant the circuit is broken,

then $$c = \text{logh}\, I$$

and $$\text{logh}\,\frac{i}{I} = -\frac{R}{L}t$$

and $$i = Ie^{-\frac{R}{L}t}$$

I is, of course, equal to $\frac{E}{R}$ of the unbroken circuit.

EXAMPLE 9

LEAKS

Cases in which the rate of escape is proportional to the quantity present or remaining.

If q = quantity present at the time t

then $$\frac{dq}{dt} = -kq \quad (k \text{ is a constant})$$

$$\int \frac{dq}{q} = -k \int dt$$

$$\text{logh } q = -kt + C$$

If Q is the quantity when $t = 0$, $C = \text{logh } Q$

and $$\text{logh } q = -kt + \text{logh } Q$$

from which $$q = Qe^{-kt}.$$

Worked Example. An electric condenser leaks such that the rate of loss is proportional to the quantity contained. At a given instant the quantity remaining is 20 coulombs, but after 60 secs. it is only 18 coulombs. Find (i) the quantity remaining at the end of the next 30 secs., and the time to lose half the charge, i.e. 10 coulombs.

From $$q = Qe^{-kt}$$

$$18 = 20\, e^{-60k}$$

$$k = \frac{\log 20 - \log 18}{60 \log e}$$

$$= \frac{\cdot 0457}{26\cdot 058}$$

$$= \cdot 001758.$$

$\therefore$ $$q = Qe^{-\cdot 001758t}.$$

To find the quantity (q) remaining after a further 30 secs., i.e. 90 secs. from $t = 0$,

$$q = 20\, e^{-\cdot 15822}$$

(i) $$= 17\cdot 07 \text{ coulombs}.$$

To find the time for the charge to fall to half, we have:

$$e^{-\cdot 001758t} = \cdot 5$$

$$t = \frac{\log \cdot 5}{\cdot 001758 \log e}$$

$$= \frac{-\cdot 3010}{-\cdot 0007638}$$

(ii) $$= 394 \text{ secs.}$$

EXAMPLE 10

It is shown in the study of engineering that the tension (T) in a belt passing over a pulley and the angle of lap (θ) are so related that the rate of change in tension with the angle is proportional to the tension.

i.e. $\dfrac{dT}{d\theta} = \mu T$ where μ is the coefficient of friction.

$$\therefore \quad \frac{dT}{T} = \mu \, d\theta$$

and by integration, logh $T = \mu\theta + C$.

If T_o is the tension where $\theta = 0$, i.e. at the slack side of the belt, then $C = \text{logh}\ T_o$,

and $\therefore$ $\quad \text{logh}\ T = \mu\theta + \text{logh}\ T_o$

or $\quad T = T_o e^{\mu\theta}$.

As an exercise, calculate T for a rope hanging over a pole for the case in which $\mu = 0{\cdot}5$, $T_o = 1$ lb. and $\theta = \pi$ radians. What is T if the rope is wrapped round so that θ is 3π?

EXAMPLE 11

The pressure of a fluid varies with the density, and it is well known that the density of the atmosphere decreases with increase in altitude.

The relation is that the rate of change of pressure with altitude is proportional to the density and therefore to the pressure, provided the temperature is constant.

If p is the pressure at altitude h

then $\dfrac{dp}{dh} = -kp$, where k, is a constant.

$$\therefore \quad \int \frac{dp}{p} = -\int k dh$$

$$\text{logh}\ p = -kh + C$$

If p_o is the pressure at $h = 0$ then

$$C = \operatorname{logh} p_o$$

$$\therefore \quad \operatorname{logh} p = -kh + \operatorname{logh} p_o$$

or

$$p = p_o e^{-kh}.$$

Exercise: If the pressure is 30 in. of mercury at sea-level and 20 in. at 10,000 ft. altitude, what would it be at 5,000 ft., the temperature being considered constant?

EXAMPLE 12

Imagine a moving body to be subjected to a retarding force proportional to its velocity. Find (i) its velocity at any instant, and (ii) the distance travelled in any given interval of time.

Let M be the mass of the body and v its velocity at any instant.

The retarding force is $-kv$ and by elementary mechanics is known to equal the product of the mass and the acceleration.

i.e.

$$M\frac{dv}{dt} = -kv$$

or

$$\frac{dv}{v} = -\frac{k}{M}dt$$

$$= -K dt \text{ where } K = \frac{k}{M}$$

Integrating with respect to t

$$\operatorname{logh} v = -Kt + C$$

If at $t = 0$ the velocity is V_o then $C = \operatorname{logh} V_o$

and $\therefore$ $\operatorname{logh} v - \operatorname{logh} V_o = -Kt$

or

$$\operatorname{logh} \frac{v}{V_o} = -Kt$$

or (i)

$$v = V_o e^{-Kt}$$

This gives the velocity at any instant.

Let S be the distance travelled in time t

then $$S = \int v dt$$

$$= \int V_o e^{-Kt} dt \qquad \text{from (i)}$$

$$= -\frac{V_o e^{-Kt}}{K} + C$$

If $S = 0$ when $t = 0$, then $C = \frac{V_o}{K}$

and (ii) $$S = \frac{V_o}{K}(1 - e^{-Kt})$$

This gives the distance travelled in the interval 0 to t.

EXERCISES

1. A ship of mass 800 tons gradually comes to rest. If the resistance of the water varies as the speed of the ship, and the ship is observed to travel 100 ft. in 30 sec. and 140 ft. in the next 60 sec., find when the velocity is 1 ft. per sec. and what the retarding force is at that instant.
2. Work out the relations corresponding to (i) and (ii) when the resisting force varies as the square of the velocity.

§2. Plot the graph of $\sin x$ for, say, three cycles (i.e. for $x = 0$ to 6π). Calculate values of $e^{-\frac{x}{10}}$ for convenient values of x, e.g. $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}$, etc., and multiply the ordinates of $\sin x$ by these values, thus obtaining the graph of $e^{-\frac{x}{10}} \sin x$.

It will be observed that the ordinates are gradually reduced by introducing the factor $e^{-\frac{x}{10}}$.

The result suggests the damping of waves; that is, the gradual decrease in amplitude.

§3. The equation to the rectangular hyperbola is

$$xy = k \text{ or } y = \frac{k}{x}$$

The area bounded by the curve is given generally by

$$\int y\,dx = \int \frac{k}{x}dx$$

$$= k \operatorname{logh} x + C.$$

Between ordinates at x_1 and x the area is $k \operatorname{logh} \frac{x_2}{x_1}$.

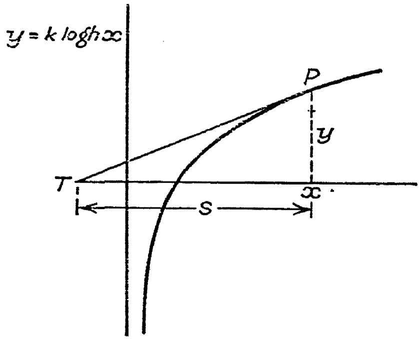

FIG. 3

§4. *The sub-tangent of the logarithmic curve* $y = k \operatorname{logh} x$

$$y = k \operatorname{logh} x$$

$$\therefore \quad \frac{dy}{dx} = \frac{k}{x}$$

Referring to figure (3) let $s =$ the length of the sub-tangent; then at any value of x,

$$\frac{y}{s} = \frac{dy}{dx} = \frac{k}{x}$$

$$s = \frac{xy}{k}$$

$$= x \operatorname{logh} x.$$

At $x = e$, $s = e \operatorname{logh} e = e$; i.e. the tangent passes through the origin in this case.

EXERCISES

1. If $r = ae^{k\theta}$, show that $\frac{dr}{d\theta} = kr$, and interpret the result.

2. On squared paper set out a number of radii making angles $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, etc., to 3π radians, with the horizontal. Make these radii of length r calculated from the equation,

$$r = ae^{k\theta}$$

in which $a = \cdot 2$, $k = \frac{1}{4}$, and θ has the above values.

Join the ends by a continuous curve thereby obtaining a spiral known as the equiangular spiral.

The area of the sector limited by two radii at angles θ_1 and θ_2 is $\frac{1}{2}\int r^2 d\theta$. Find the area for $\theta_2 = \pi$ and $\theta_1 = 0$, and for $\theta_2 = \frac{5\pi}{2}$ and $\theta_1 = 2\pi$. Verify the results by counting the squares.

CHAPTER III

HYPERBOLIC FUNCTIONS

§1. The hyperbolic functions are formed from the exponential functions e^x and e^{-x}.

Thus

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{etc.}$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \text{etc.}$$

By addition and division by 2,

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \text{etc.} \quad . \quad (1)$$

and by subtraction and division by 2,

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \text{etc.} \quad . \quad (2)$$

Because of similarities between these and trigonometrical functions, which will be appreciated later,

$$\frac{e^x + e^{-x}}{2} \text{ is called } \mathbf{cosh}\, x \qquad (3)$$

and $$\frac{e^x - e^{-x}}{2} \text{ is called } \mathbf{sinh}\, x \quad . \qquad (4)$$
(pronounced shine)

Also, $$\frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ is called } \mathbf{tanh}\, x \quad . \qquad (5)$$
(pronounced tank)

The graphs of $\cosh x$ and $\sinh x$ are shown in Fig. 4.

§2. The following relations are important and in some cases are similar to those met in trigonometry.

$$\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x \qquad (6)$$

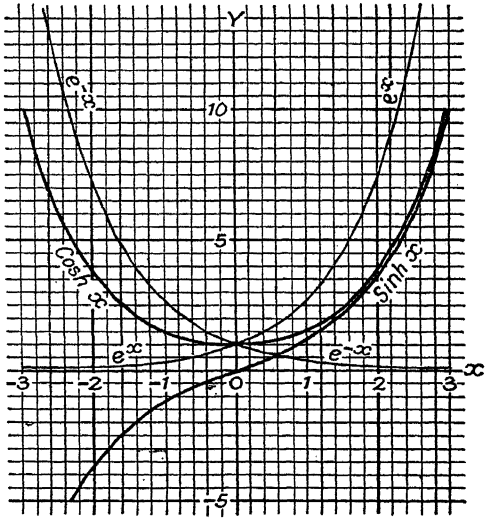

FIG. 4

Graphs of e^x, e^{-x}, cosh x, and sinh x

$$\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = e^{-x} \tag{7}$$

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= e^x \times e^{-x} = e^0$$

i.e. $\cosh^2 x - \sinh^2 x = 1$ . . . . . . (8)

Hence $\cosh x = \sqrt{\sinh^2 x + 1}$ . . . (9)

$\sinh x = \sqrt{\cosh^2 x - 1}$ . . . . (10)

Again,

$(\cosh x + \sinh x)(\cosh y + \sinh y) = e^x \times e^y = e^{(x+y)}$

$(\cosh x - \sinh x)(\cosh y - \sinh y) \qquad = e^{-(x+y)}$

Multiplying out,

$$\cosh x \cosh y + \sinh x \sinh y + \cosh x \sinh y + \sinh x \cosh y = e^{(x+y)}$$

$$\cosh x \cosh y + \sinh x \sinh y - \cosh x \sinh y - \sinh x \cosh y = e^{-(x+y)}$$

By addition,

$$\cosh x \cosh y + \sinh x \sinh y = \frac{e^{(x+y)} + e^{-(x+y)}}{2} = \cosh(x+y) \quad . \quad . \quad . \quad . \quad (11)$$

By subtraction,

$$\cosh x \sinh y + \sinh x \cosh y = \frac{e^{(x+y)} - e^{-(x+y)}}{2} = \sinh(x+y) \quad . \quad . \quad . \quad . \quad (12)$$

If in (11) and (12), $y = x$, then

$$\cosh 2x = \cosh^2 x + \sinh^2 x \quad . \quad . \quad .$$

$$= 2\cosh^2 x - 1 \text{ or } 2\sinh^2 x + 1 \quad . \quad (13)$$

and $$\sinh 2x = 2\cosh x \sinh x \quad . \quad . \quad . \quad . \quad (14)$$

The reciprocals correspond to those in trigonometry, thus:

$$\text{cosech} = \frac{1}{\sinh}, \quad \text{sech} = \frac{1}{\cosh}, \quad \coth = \frac{1}{\tanh}$$

The following are easily proved:

$$\text{sech}\, x = \sqrt{1 - \tanh^2 x} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\text{cosech}\, x = \sqrt{\coth^2 x - 1} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

§3. **Inverse Hyperbolic Functions.** The inverse hyperbolic functions can be expressed in terms of logarithmic functions.

If $\cosh^{-1}\frac{x}{a} = y$, then $\cosh y = \frac{x}{a}$

and since $\sinh y = \sqrt{\cosh^2 y - 1}$

$$\sinh y = \pm \frac{\sqrt{x^2 - a^2}}{a}$$

But $\cosh y + \sinh y = e^y$

$$\frac{x \pm \sqrt{x^2 - a^2}}{a} = e^y$$

or
$$y = \operatorname{logh} \frac{x \pm \sqrt{x^2 - a^2}}{a}$$

i.e.
$$\boldsymbol{\cosh^{-1} \frac{x}{a} = \operatorname{logh} \frac{x \pm \sqrt{x^2 - a^2}}{a}} \qquad (17)$$

Similarly,

$$\boldsymbol{\sinh^{-1} \frac{x}{a} = \operatorname{logh} \frac{x +^* \sqrt{x^2 + a^2}}{a}} \qquad (18)$$

If $\tanh^{-1}\frac{x}{a} = y$, then $\frac{x}{a} = \tanh y$

i.e.
$$\frac{x}{a} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

By proportion $\left(\frac{\text{sum}}{\text{difference}}\right)$ $\frac{a + x}{a - x} = \frac{2e^y}{2e^{-y}}$

and since $\frac{1}{e^{-y}} = e^y$

$$y = \frac{1}{2} \operatorname{logh} \frac{a + x}{a - x}$$

i.e.
$$\boldsymbol{\tanh^{-1} \frac{x}{a} = \frac{1}{2} \operatorname{logh} \frac{a + x}{a - x}} \qquad (19)$$

Similarly,

$$\boldsymbol{\coth^{-1} \frac{x}{a} = \frac{1}{2} \operatorname{logh} \frac{x + a}{x - a}} \qquad (20)$$

* Since $\sqrt{x^2 + a^2} > x$ and there are no logarithms of negative quantities the plus sign only is admissible.

EXERCISES

1. Show that the sum of the two logh values of $\cosh^{-1}\frac{x}{a}$ is 0.

2. Verify that $\coth^{-1}\frac{x}{a}$ also equals $-\frac{1}{2}\operatorname{logh}\frac{x-a}{x+a}$.

3. Establish the logarithmic form of $\tanh^{-1}\frac{x}{a}$ by multiplying the numerator and denominator of $\frac{e^y-e^{-y}}{e^y+e^{-y}}$ by e^y, and finding y.

4. Show that for $x > 7$, $\sinh x$ is nearly the same as $\cosh x$. To what decimal place are they alike when $x = 7$?

5. Show that $\cosh x + \sinh x$ is the reciprocal of $\cosh x - \sinh x$.

Differentiation and Integration of Hyperbolic Functions.

§4. If $y = \cosh x = \frac{e^x}{2} + \frac{e^{-x}}{2}$

then $$\frac{dy}{dx} = \frac{e^x}{2} - \frac{e^{-x}}{2} = \sinh x \qquad (21)$$

and $\therefore$ $$\int \sinh x\, dx = \cosh x + C \qquad (22)$$

If $y = \sinh x = \frac{e^x}{2} - \frac{e^{-x}}{2}$

then $$\frac{dy}{dx} = \frac{e^x}{2} + \frac{e^{-x}}{2} = \cosh x \qquad (23)$$

and $\therefore$ $$\int \cosh x\, dx = \sinh x + C \qquad (24)$$

If $y = \tanh x = \frac{\sinh x}{\cosh x}$

then $$\frac{dy}{dx} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x \qquad (25)$$

and $\therefore$ $$\int \operatorname{sech}^2 x\, dx = \tanh x + C \qquad (26)$$

Similarly, $$\frac{d}{dx}\coth x = -\operatorname{cosech}^2 x \qquad (27)$$

and $$\int \operatorname{cosech}^2 x\, dx = -\coth x + C \qquad (28)$$

If $y = \cosh^{-1}\frac{x}{a}$ then $x = a\cosh y$

and
$$\frac{dx}{dy} = a \sinh y$$
$$= a\sqrt{\cosh^2 y - 1}$$
$$= \sqrt{x^2 - a^2}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - a^2}} \quad . \quad . \quad . \qquad (29)$$

and
$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} = \operatorname{logh} x \pm \sqrt{x^2 - a^2} \qquad (30)$$

Result (30) should be contrasted with
$$\int \frac{dx}{\quad} = \sin^{-1}\frac{x}{\quad}$$

If $y = \sinh^{-1}\frac{x}{a}$, then $x = a \sinh y$

and
$$\frac{dx}{dy} = a \cosh y$$
$$= a\sqrt{\sinh^2 y + 1}$$
$$= \sqrt{x^2 + a^2}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + a^2}} \quad . \quad . \qquad (31)$$

and
$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} = \operatorname{logh}\frac{x + \sqrt{x^2 + a^2}}{a} \qquad (32)$$

If $y = \tanh^{-1}\frac{x}{a}$, then $x = a \tanh y$

and
$$\frac{dx}{dy} = a \operatorname{sech}^2 y$$
$$= a(1 - \tanh^2 y)$$
$$\frac{a^2 - x^2}{a}$$
$$\frac{dy}{dx} = \frac{a}{a^2 - x^2} \qquad (33)$$

and $$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1}\frac{x}{a} = \frac{1}{2a} \operatorname{logh} \frac{a + x}{a - x} \tag{34}$$

Similarly, $$\frac{d}{dx} \coth^{-1}\frac{x}{a} = -\frac{a}{x^2 - a^2} \tag{35}$$

and $$\int \frac{dx}{x^2 - a^2} = -\frac{1}{a} \coth^{-1}\frac{x}{a} = \frac{1}{2a} \operatorname{logh} \frac{x - a}{x + a} \tag{36}$$

NOTE: In (33) $x^2 < a^2$, in (35) $x^2 > a^2$.

Since the differential coefficient of $x \pm c$ is the same as that of x it follows that

(i) $$\int \frac{dx}{\sqrt{(x \pm c)^2 + a^2}} \quad \sinh^{-1} \frac{x \pm c}{}$$

(ii) $$\int \frac{dx}{\sqrt{(x \pm c)^2 - a^2}} = \cosh^{-1} \frac{x \pm c}{a}$$

(iii) $$\int \frac{dx}{a^2 - (x \pm c)^2} = \frac{1}{a} \tanh^{-1} \frac{x \pm c}{a}$$

§ 5. $$\int \tanh x \, dx = \int \frac{\sinh x \, dx}{\cosh x}$$

Let $u = \cosh x$, then $du = \sinh x \, dx$,

and $$\int \tanh x \, dx = \int \frac{du}{u} = \operatorname{logh} u = \operatorname{logh} \cosh x \tag{37}$$

Similarly, $$\int \coth x \, dx = \operatorname{logh} \sinh x \tag{38}$$

$$\int \frac{dx}{\sinh x} = \int \frac{dx}{2 \cosh \frac{x}{2} \sinh \frac{x}{2}}$$

$$= \int \frac{\operatorname{sech}^2 \frac{x}{2} \, dx}{2 \cosh \frac{x}{2} \sinh \frac{x}{2} \operatorname{sech}^2 \frac{x}{2}}$$ [multiplying numerator and denominator by $\operatorname{sech}^2 \frac{x}{2}$]

$$= \int \frac{2d\left(\tanh \frac{x}{2}\right)}{2 \tanh \frac{x}{2}} \qquad \left[\begin{array}{l} \text{Since } \dfrac{d}{dx} \cdot \tanh \dfrac{x}{2} = \frac{1}{2} \operatorname{sech}^2 \dfrac{x}{2} \\ \text{and } \operatorname{sech}^2 \dfrac{x}{2} = \dfrac{1}{\cosh^2 \frac{x}{2}} \end{array}\right.$$

$$= \text{logh} \tanh \frac{x}{2} \quad . \quad . \quad . \quad . \qquad (39)$$

$$\int \frac{dx}{\cosh x} = \int \frac{2\,dx}{e^x + e^{-x}} = \int \frac{2e^x dx}{e^{2x} + 1} \left[\begin{array}{c}\text{multiplying numer-} \\ \text{ator and denomin-} \\ \text{ator by } e^x. \end{array}\right]$$

Put $u = e^x$, then $du = e^x dx$, and $u^2 = e^{2x}$

$$\text{and } \therefore \int \frac{dx}{\cosh x} = \int \frac{2du}{u^2 + 1} = 2 \tan^{-1} u = 2 \tan^{-1} e^x \quad (40)$$

$$\text{e.g.} \qquad \int_0^{\frac{1}{2}} \frac{dx}{\cosh x} = 2\left[\tan^{-1} e^{\frac{1}{2}} - \tan^{-1} e^0\right]$$

$$= 2\left[\tan^{-1} 1{\cdot}6487 - \tan^{-1} 1\right]$$

$$= 2\left[1{\cdot}0356 - {\cdot}7854\right] \text{ radians}$$

$$= {\cdot}5004 \text{ radians}$$

EXERCISES

1. Find the three successive differential coefficients of $\cosh^{-1}x$ and $\sinh^{-1}x$.

2. In order to appreciate the distinction between

and

take $a = 4$ and plot $\frac{1}{\sqrt{a^2 - x^2}}$ and $\frac{1}{\sqrt{x^2 - a^2}}$ as shown in Fig. 5.

Estimate the area between, say $x = 0$ and $x = 3$ and between $x = 5$ and $x = 8$, and compare the results with the values of the integrals. Show also that $\int_0^4 \frac{dx}{\sqrt{a^2 - x^2}} = \frac{\pi}{2}$.

3. Find

(i) $\int \frac{dx}{\sqrt{(x + 3)^2 + 4}}$ (ii) $\int \frac{dx}{\sqrt{(x + 3)^2 - 4}}$

(iii) $\int \frac{dx}{\sqrt{4 - (x + 3)^2}}$

4. Using the method indicated in Exercise 3, find

(i) $\displaystyle\int \frac{dx}{\sqrt{x^2 + 6x + 13}}$ (ii) $\displaystyle\int \frac{}{\sqrt{x^2 + 4x + 13}}$

(iii) $\displaystyle\int \frac{dx}{\sqrt{x^2 - 4x + 13}}$ (iv) $\displaystyle\int \frac{dx}{\sqrt{x^2 + 4x + 9}}$

(v) $\displaystyle\int \frac{dx}{\sqrt{x^2 - 4x - 1}}$ (vi) $\displaystyle\int \frac{dx}{\sqrt{x^2 + 5x + 6}}$

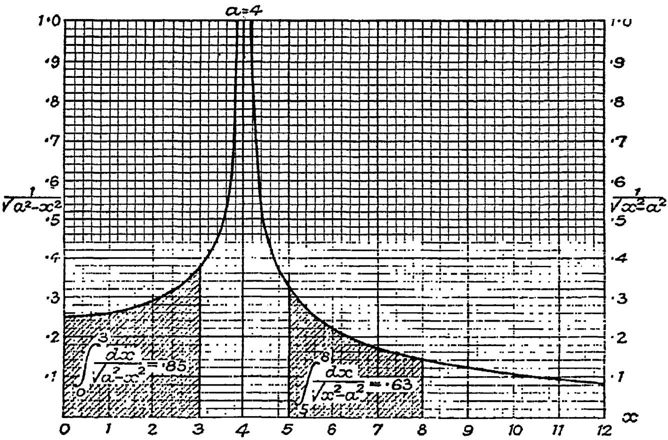

FIG. 5

Graphs illustrating $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}}$ and $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}}$

5. Show that

$$\int (\cosh x + \sinh x)^n \, dx = \frac{e^{nx}}{n}$$

6. Show that

$$\int \frac{\sinh x \, dx}{\cosh x + \sinh x} = \tfrac{1}{2}\int (1 - e^{-2x}) \, dx = \tfrac{1}{2}x + \tfrac{1}{4}e^{-2x}$$

7. By putting $x = \cosh u$, show that

$$\int \frac{dx}{x + \sqrt{x^2 - 1}} = \tfrac{1}{2}\cosh^{-1} x + \tfrac{1}{2}x\,(x - \sqrt{x^2 - 1})$$

§6. * THREE IMPORTANT INTEGRALS COMPARED

(i) $\int \sqrt{a^2 - x^2}\, dx$

Let $\frac{x}{a} = \cos\theta$, then $\sqrt{a^2 - x^2} = a\sin\theta$

and $dx = -a\sin\theta\ d\theta$

$$\text{(i)} = -\int a^2 \sin^2\theta\, d\theta$$

$$= a^2 \int \frac{\cos 2\theta - 1}{2} d\theta$$

$$= \frac{a^2}{2}\left(\frac{\sin 2\theta}{2} - \theta\right)$$

$$= \frac{a^2}{2}\left(\sin\theta\ \cos\theta - \cos^{-1}\frac{x}{a}\right)$$

$$= \frac{x}{2}\sqrt{a^2 - x^2} - \frac{a^2}{2}\cos^{-1}\frac{x}{a}\dagger$$

† In calculating the value of this expression remember that for $x = 0$, $\cos^{-1}\frac{x}{a} = \frac{\pi}{2}$. The expression is then seen to agree with the more commonly given

$$\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

(ii) $\int \sqrt{x^2 - a^2\, dx}$

Let $\frac{x}{a} = \cosh u$, then $\sqrt{x^2 - a^2} = a\sinh u$

and $dx = a\sinh u\, du$

$$\text{(ii)} = \int a^2 \sinh^2 u\, du$$

$$= a^2 \int \frac{\cosh 2u - 1}{2} du$$

$$= \frac{a^2}{2}\left(\frac{\sinh 2u}{2} - u\right)$$

$$= \frac{a^2}{2}\left(\sinh u\ \cosh u - \cosh^{-1}\frac{x}{a}\right)$$

$$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$$

(iii) $\int \sqrt{x^2 + a^2}\, dx$

Let $\frac{x}{a} = \sinh u$,

then $\sqrt{x^2 + a^2} = a\cosh u$

and $dx = a\cosh u\, du$

$$\text{(iii)} = \int a^2 \cosh^2 u\, du$$

$$= a^2 \int \frac{\cosh 2u + 1}{2} du$$

$$= \frac{a^2}{2}\left(\frac{\sinh 2u}{2} + u\right)$$

$$= \frac{a^2}{2}\left(\sinh u\ \cosh u + \sinh^{-1}\frac{x}{a}\right)$$

$$= \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$$

EXERCISES

Evaluate—

1. $\int_0^2 \sqrt{4 - x^2}\, dx$ 2. $\int_0^1 \sqrt{8 - 3x^2}\, dx$ 3. $\int \sqrt{x^2 - 4}\, dx$

4. $\int \sqrt{x^2 + 25}\, dx$ 5. $\int_0^8 \sqrt{3x^2 + 12}\, dx$

6. Show that the area bounded by the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

the axis of x and any ordinate y is

$$\left(\frac{xy}{2} - \frac{ab}{2} \cosh^{-1} \frac{x}{a}\right)$$

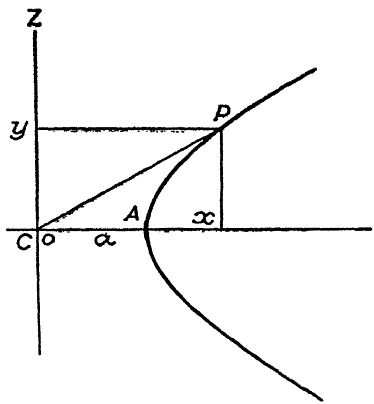

FIG. 6

§7. **Area of Hyperbolic Sector.*** In Fig. 6 CPA is a hyperbolic sector, the equation of the hyperbola being

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

CZ (axis of y) the directrix, CA being equal to a. (b is the semi-conjugate axis.)

From the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$y = \frac{b}{a}\sqrt{x^2 - a^2}$$

The area of figure $APx = \int y\,dx$

$$= \frac{b}{a}\int \sqrt{x^2 - a^2}\,dx$$

$$= \frac{b}{a}\left(\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}\right)$$

$$= \frac{xy}{2} - \frac{ab}{2}\cosh^{-1}\frac{x}{a}$$

Sector $CPA = \triangle\, CPx -$ figure APx

$$= \frac{xy}{2} - \left(\frac{xy}{2} - \frac{ab}{2}\cosh^{-1}\frac{x}{a}\right)$$

$$= \frac{ab}{2}\cosh^{-1}\frac{x}{a}$$

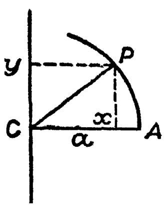

FIG. 7

This result accounts for the name hyperbolic functions.

Take for comparison the circle $x^2 + y^2 = a^2$. (Fig. 7.)

The area of the sector CPA of this circle is $\frac{1}{2}\, CA \times PA$.

If x is the abscissa of P, $\cos \widehat{ACP} = \frac{x}{a}$ or $\widehat{ACP} = \cos^{-1}\frac{x}{a}$ and arc $PA = a\cos^{-1}\frac{x}{a}$.

Area of circular sector CPA $\quad {}^{-1}\frac{x}{\ }$.

The similarity between the two results is obvious. If

in the first a and b are equal, as in the case of the rectangular hyperbola, the similarity is even more striking.

Further, the co-ordinates of P in the case of the hyperbola may be written $a \cosh u$ and $b \sinh u$ where $\cosh u = \frac{x}{a}$ and in the case of the circle as $a \cos \theta$ and $a \sin \theta$ where $\cos \theta = \frac{x}{a}$.

§8. **The Catenary.*** The graph of $y = \cosh x$ is a curve called the catenary (Fig. 8). It is the form of a uniform wire, rope or chain hanging freely under the action of gravity.

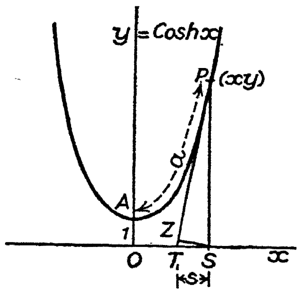

FIG. 8

Observe that its vertex A is unit distance above the origin O.

The gradient of the tangent at any point $P(xy)$ is:

$$\frac{dy}{dx} = \sinh x$$

The length of the subtangent (s) is:

$$TS = y \bigg/ \frac{dy}{dx} = \frac{\cosh x}{\sinh x} = \coth x$$

The length of the tangent is:

$$PT = \sqrt{y^2 + s^2} = \sqrt{\cosh^2 x + \coth^2 x}$$
$$= \cosh x \coth x$$

The length of the arc, AP, (a) is

$$a = \int \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx = \int \sqrt{1 + \sinh^2 x} \,.\, dx$$
$$= \int \cosh x \, dx = \sinh x + C$$

If measured from the vertex A, a is 0 for $x = 0$, then $C = 0$ and $a = \sinh x$.

If from the foot of the ordinate PS, SZ is drawn perpendicular to PT then—

$$\frac{PZ}{PS} = \frac{PS}{PT} \text{ and } \therefore PZ = \frac{PS^2}{PT} = \frac{\cosh^2 x}{\cosh x \coth x} = \sinh x = a$$

i.e. PZ is equal to the length of the arc AP.

It is easily proved that SZ is always unity and therefore equal to OA.

EXERCISES

1. Show that the area under the catenary $y = \cosh x$ is $\sinh x$ and that the volume of the solid of revolution about the axis of x is $\frac{\pi}{4}(\sinh 2x + 2x)$.

2. Verify that $\sinh x$ and $\cosh x$ are solutions of the differential equation $\frac{d^2y}{dx^2} = y$.

3. Find the length of the arc of the catenary $y = \cosh x$ from $x = +3$ to $x = -3$.

4. A body moves so that its distance from a fixed point is equal to $\cosh kt$, where t represents time. What are its velocity and acceleration if its position is 10 at $t = 1$?

CHAPTER IV

OTHER SERIES FORMS

§1. To Represent Sine and Cosine in Series Form. The method is almost the same as that on page **1**.

Let $\sin x = a + bx + cx^2 + dx^3 + \text{etc.}$ (1)

Since this must hold for all values of x it must hold for $x = 0$. But $\sin 0 = 0$, $\therefore a = 0$.

Now differentiate both sides of (1), then

$$\cos x = b + 2cx + 3dx^2 + 4ex^3 + \text{etc.} \quad . \qquad (2)$$

Again, this must hold for $x = 0$. But $\cos 0 = 1 \therefore b = 1$.

Differentiate again, then

$$-\sin x = 2c + 2 \times 3dx + 3 \times 4ex^2 + \text{etc.} \qquad (3)$$

which must hold for $x = 0$. $\therefore c = 0$.

Continuing the differentiation,

$$-\cos x = 2 \times 3d + 2 . 3 . 4ex + 3 . 4 . 5fx^2 \qquad (4)$$

from which since $-\cos 0 = -1$, $d = -\dfrac{1}{3!}$.

Similarly, $e = 0$, and $f = +\dfrac{1}{5!}$,

and $g = 0$, and $h = -\dfrac{1}{7!}$, etc.

Hence from (1)

$$\sin \mathbf{x} = \mathbf{x} - \frac{\mathbf{x}^3}{3!} + \frac{\mathbf{x}^5}{5!} - \frac{\mathbf{x}^7}{7!} \text{ etc.} \quad . \qquad (5)$$

and from (2)

$$\cos \mathbf{x} = 1 - \frac{\mathbf{x}^2}{2!} + \frac{\mathbf{x}^4}{4!} - \frac{\mathbf{x}^6}{6!} + \text{etc.} \quad . \qquad (6)$$

4—(5731)

EXERCISES

1. Verify results (5) and (6) by taking x to be values such as $\frac{\pi}{6}$, $\frac{\pi}{4}$, etc., and comparing with the values given in tables. (Observe that x is in radians, not degrees.)

2. *Approximate evaluation of* π. The series form of $\cos x$ may be used to find the approximate value of π. Take a value of x of which the cosine is known, e.g. $\frac{\pi}{6}$, and use the first three terms of the series (6), thereby obtaining a double quadratic equation in π which is easily solved, giving $\pi = 3{\cdot}142$.

3. Show that $\sin^2 x = x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6$, etc.

4. Obtain the series for $\cos x$ by differentiating the series for $\sin x$.

§2. **The Logarithmic Series.** If we try to establish a series form of $\operatorname{logh} x$ by the method used for $\sin x$, difficulty is met immediately, since $\operatorname{logh} 0$ is $-\infty$. On the other hand $\operatorname{logh}(1 + x)$ is easily expanded in series form, thus—

$$\text{Let } \operatorname{logh}(1 + x) = a + bx + cx^2 + dx^3 + ex^4 \text{ etc.} \quad . \quad (7)$$

then for $x = 0$ we have, $\operatorname{logh} 1 = a$ and since $\operatorname{logh} 1 = 0$, $a = 0$.

The first differentiation of (7) gives—

$$\frac{1}{1 + x} = b + 2cx + 3dx^2 + 4ex^3 + \text{etc.} \quad . \quad (8)$$

from which, by putting $x = 0$, it follows that $b = 1$.

By further successive differentiations the remaining coefficients of (7) can be determined, but the following is quicker—

$$\text{By division, } \frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 \text{ etc.} \quad . \quad (9)$$

By comparing series (8) with series (9) term by term, it is immediately evident that—

$$b = 1,\ c = -\tfrac{1}{2},\ d = +\tfrac{1}{3},\ e = -\tfrac{1}{4}, \text{ etc.}$$

and therefore from (7)

$$\operatorname{logh}(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \text{ etc.} \quad . \quad (10)$$

By putting $x = 1, 2, 3$, etc., respectively in this series logarithms to base e can be obtained, but the calculation is very tedious. The following method is quicker.

If we substitute $-x$ for x in the series form of

$$\text{logh}\,(1 + x)$$

we obtain—

$$\text{logh}\,(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\text{ etc.} \quad . \qquad (11)$$

and subtracting logh $(1 - x)$ from logh $(1 + x)$, remembering that the difference in logarithms is the logarithm of the quotient of the quantities, we obtain the more rapidly converging series,

$$\text{logh}\,\frac{1 + x}{1 - x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5}\text{ etc.}\right) \quad . \qquad . \; (12)$$

If we put $x = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, the values of $\dfrac{1 + x}{1 - x}$ are respectively 2, 3, 5, 7, and we can now calculate the logarithms of these numbers from series (12).

From these logarithms the logarithms of other numbers can be found; e.g. logh 10 from the sum of logh 2 and logh 5.

As exercises, (i) establish the series forms of logh $(1 + x)$ and logh $(1 - x)$ by successive differentiation.

(ii) Show that the first six successive differential coefficients of logh $\cos x$ are, respectively,

$$-Z,\ -u,\ -2uZ,\ -2u\,(3Z^2 + 1),\ -8uZ\,(3Z^2 + 2),$$
$$-8u\,(15Z^4 + 15Z^2 + 2),\text{ where } Z = \tan x \text{ and } u = \sec^2 x,$$

and that $\text{logh}\cos x = -\dfrac{x^2}{2} - \dfrac{x^4}{12} - \dfrac{x^6}{45}$, etc.

and $\qquad \text{logh}\sec x = \dfrac{x^2}{2} + \dfrac{x^4}{12} + \dfrac{x^6}{45}$, etc.

(iii) Show that $\text{logh}\,(1 + \sin x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{6}$, etc.

§3. *To represent* $(1 + x)^n$ *as a power series.**

Let $(1 + x)^n = a + bx + cx^2 + dx^3$ etc.

For $\quad x = 0,\ a = 1.$

By differentiation—

$$n(1 + x)^{n-1} = b + 2cx + 3dx^2 \text{ etc.}$$

For $x = 0,\ b = n$.

Continuing, the differentiation,

$$n(n-1)(1 + x)^{n-2} = 2c + 2 . 3dx + 3 . 4ex^2 + \text{etc.}$$

For $x = 0,\ c = \dfrac{n(n-1)}{2}$.

Similarly $d = \dfrac{n(n-1)(n-2)}{3!}$

$$e = \frac{n(n-1)(n-2)(n-3)}{4!} \text{etc.}$$

$$\therefore\ (1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \text{ etc.}$$

The rth term being $\dfrac{n(n-1)\ .\ .\ .\ (n-r+2)}{(r-1)!}x^{r-1}$.

This is the binomial expansion, of course.

As an exercise, expand $\left(1 + \dfrac{1}{n}\right)^n$. Confirm that if n is infinitely great the series is exponential e.

NOTE: The exponential series is frequently defined as the expansion of $\left(1 + \dfrac{x}{n}\right)^n$ to infinity.

CHAPTER V

EXPONENTIAL FORMS OF SINE AND COSINE

§1. The similarity between the series forms of sine and cosine, and the exponential series will have been observed.

Taking no account of the signs, the terms of the sine series are like the odd-power terms of the exponential series, and those of the cosine series are like the even-power terms of the exponential series.

The same likeness exists between them and the series for sinh and cosh.

By using the imaginary $\sqrt{-1}$, we can link up the series forms of sine and cosine with the exponential function. Write† j for $\sqrt{-1}$ then—

$$j^2 = -1 \qquad \text{and } j^3 = j^2 \times j = -j$$
$$j^4 = (j^2)^2 = +1 \qquad \text{and } j^5 = j^4 \times j = +j$$
$$j^6 = (j^2)^3 = -1 \qquad \text{and } j^7 = j^6 \times j = -j, \text{ etc.}$$

Hence the successive powers of j, namely—

$$j, \quad j^2, \quad j^3, \quad j^4, \quad j^5, \quad j^6, \quad j^7, \text{ etc.}$$

equal respectively, $j, -1, -j, +1, +j, -1, -j$, etc.

Substituting jx for x in the exponential series we have—

$$e^{jx} = 1 + jx + \frac{j^2x^2}{2\,!} + \frac{j^3x^3}{3\,!} + \frac{j^4x^4}{4\,!} + \frac{j^5x^5}{5\,!} + \frac{j^6x^6}{6\,!} + \text{etc.}$$

$$= 1 + jx - \frac{x^2}{2\,!} - j\frac{x^3}{3\,!} + \frac{x^4}{4\,!} + j\frac{x^5}{5\,!} - \frac{x^6}{6\,!}\text{ etc.}$$

$$= \left(1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \frac{x^6}{6\,!}\text{ etc.}\right) + j\left(x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!}\text{ etc.}\right)$$

i.e. $\qquad \mathbf{e^{jx} = \cos x + j \sin x} \qquad (1)$

† In textbooks on pure mathematics i is frequently used for $\sqrt{-1}$, but in applied science it is usual to use j.

Similarly—

$$e^{-jx} = 1 - jx + \frac{j^2x^2}{2\,!} - \frac{j^3x^3}{3\,!} + \frac{j^4x^4}{4\,!} - \frac{j^5x^5}{5\,!} + \text{etc.}$$

$$= 1 - jx - \frac{x^2}{2\,!} + \frac{jx^3}{3\,!} + \frac{x^4}{4\,!} - j\frac{x^5}{5\,!}\ \text{etc.}$$

$$= \left(1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!}\,\text{etc.}\right) - j\left(x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!}\,\text{etc.}\right)$$

i.e. $$\mathbf{e^{-jx} = \cos x - j \sin x} \quad . \quad . \quad . \quad (2)$$

By adding (1) and (2) we obtain,

$$\cos x = \frac{e^{jx} + e^{-jx}}{2} = \cosh jx \qquad (3)$$

and by subtracting,

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} = \frac{\sinh jx}{j} \quad . \quad . \quad . \quad . \quad (4)$$

Compare results (3) and (4) with

$$\cosh x = \frac{e^x + e^{-x}}{2} \text{ and } \sinh x = \frac{e^x - e^{-x}}{2}$$

By substituting jx for x in (3) and (4) we obtain,

$$\cos jx = \frac{e^{j^2x} + e^{-j^2x}}{2} \qquad \sin jx = \frac{\quad - e^{-j^2x}}{2j}$$

$$= \frac{e^{-x} + e^x}{2} \qquad \dagger = -\frac{1}{j}\,\frac{e^x - e^{-x}}{2}$$

i.e. $\cos jx = \cosh x$ (5) i.e. $\sin jx = j \sinh x$. (6)

Also, by adding and subtracting (5) and j times (6) we have—

$$\cos jx \pm j \sin jx = \cosh x \pm j^2 \sinh x$$
$$= \cosh x \mp \sinh x$$

i.e. $$\cos jx \pm j \sin jx = e^{\mp x} \quad . \quad . \quad . \quad . \quad (7)$$

$$\dagger\ -\frac{1}{j} = -\frac{j}{j^2} = -\frac{j}{-1} = +j.$$

The following are useful in electrical engineering—

$$\sinh(a \pm jb) = \sinh a \cosh jb \pm \sinh jb \cosh a$$
$$= \sinh a \cos b \pm j \sin b \cosh a \quad . \quad . \quad (8)$$

$$\cosh(a \pm jb) = \cosh a \cosh jb \pm \sinh a \sinh jb$$
$$= \cosh a \cos b \pm j \sinh a \sin b \quad . \quad . \quad (9)$$

§2. **De Moivre's Theorem.** We have seen that,

$$\cos x + j \sin x = e^{jx}$$

Raising each side to the nth power we obtain,

$$(\cos x + j \sin x)^n = e^{jnx}$$

But

$$e^{jnx} = 1 + jnx + \frac{j^2(nx)^2}{2!} + \frac{j^3(nx)^3}{3!} + \frac{j^4(nx)^4}{4!} + \frac{j^5(nx)^5}{5!} + \frac{j^6(nx)^6}{6!} + \frac{j^7(nx)^7}{7!} \text{ etc.}$$

$$= 1 + jnx - \frac{(nx)^2}{2!} - j\frac{(nx)^3}{3!} + \frac{(nx)^4}{4!} + j\frac{(nx)^5}{5!} - \frac{(nx)^6}{6!} - j\frac{(nx)^7}{7!} \text{ etc.}$$

$$= \cos nx + j \sin nx$$

Hence

$$(\cos x + j \sin x)^n = \cos nx + j \sin nx \quad . \quad (10)$$

This is De Moivre's theorem which gives the power of a trigonometrical function of x in terms of functions of the index multiple of x.

Similarly,

$$(\cos x - j \sin x)^n = e^{-jnx}$$
$$= \cos nx - j \sin nx. \quad . \quad . \quad (11)$$

By the application of De Moivre's theorem functions of multiples of x can be readily expressed in terms of functions of x, thus—

$$(\cos x + j \sin x)^3 = \cos 3x + j \sin 3x.$$

By expansion of $(\cos x + j \sin x)^3$ we have

$$\cos^3 x + 3\cos^2 x \,.\, j \sin x + 3 \cos x\, j^2 \sin^2 x + j^3 \sin^3 x$$
$$= \cos 3x + j \sin 3x$$

and since $j^2 = -1$ and $j^3 = -j$,

$$\dagger\ \cos^3 x - 3 \cos x \sin^2 x + j(3 \cos^2 x \sin x - \sin^3 x)$$
$$= \cos 3x + j \sin 3x.$$

Equating real to real and then imaginary to imaginary parts, we have

$$\cos 3x = \cos^3 x - 3 \cos x \sin^2 x = 4 \cos^3 x - 3 \cos x$$

and $\sin 3x = 3 \cos^2 x \sin x - \sin^3 x = 3 \sin x - 4 \sin^3 x.$

These are two well-known identities.

As an exercise, find $\cos 2x$, $\sin 2x$, $\cos 4x$ and $\sin 4x$ by De Moivre's theorem.

§3.* Many other functions can be put in series form by the method used in the foregoing pages.

A few examples follow.

* *Series form of tan x.* Let

$$\tan x = a + bx + cx^2 + dx^3 + ex^4 \text{ etc.} \qquad \text{(i)}$$

Putting $x = 0$, since $\tan 0 = 0$ it follows that $a = 0$.

From the first differentiation we have, since

$$\sec^2 x = \tan^2 x + 1,$$
$$\tan^2 x + 1 = b + 2cx + 3dx^2 + 4ex^3 + \text{etc.} \qquad \text{(ii)}$$

For $x = 0$, we have, $b = 1$.

From the second differentiation,

$$2 \tan x \sec^2 x = 2c + 3!dx + 3 \,.\, 4ex^2 \text{ etc.} \qquad \text{(iii)}$$

For $x = 0$, we have $c = 0$.

From the third differentiation,

$$2 \,.\, 3 \tan^2 x \sec^2 x + 2 \sec^2 x = 3!d + 4!ex$$
$$+ 3 \,.\, 4 \,.\, 5fx^2 \text{ etc.} \qquad \text{(iv)}$$

† In such complex equations the real part of one side is equal to the real part of the other, and likewise the imaginary parts.

For $x = 0$, we obtain, since $2 \sec^2 x = 2(\tan^2 x + 1)$

$$3\,!d = 2$$

$$\therefore \quad d = \tfrac{1}{3}$$

Further steps give $e = 0$, $f = \dfrac{2}{15}$, $g = 0$, $h = \dfrac{17}{315}$ etc.

The series is,

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \text{etc.}$$

* *To represent* (1) $\sin^{-1}x$ *and* (2) $\tan^{-1}x$ *as power series of* x.

(1) $$\sin^{-1}x = a + bx + cx^2 + dx^3 \text{ etc.} \quad . \qquad \text{(i)}$$

For $x = 0$ $a = 0$ (or π).

By first differentiation,

$$\frac{1}{\sqrt{1-x^2}} \text{ or } (1-x^2)^{-\frac{1}{2}} = b + 2cx + 3dx^2 + 4ex^3 + \text{etc.} \qquad \text{(ii)}$$

For $x = 0$, $b = 1$.

By further differentiation,

$$x(1-x^2)^{-\frac{3}{2}} = 2c + 3\,!dx + 3 \,.\, 4ex^2 + \text{etc.} \quad . \qquad \text{(iii)}$$

For $x = 0$, $c = 0$.

Continuing differentiation,

$$(1-x^2)^{-\frac{3}{2}} + 3x^2(1-x^2)^{-\frac{5}{2}} = 3\,!d + 4\,!ex + 3 \,.\, 4 \,.\, 5fx^2 + \text{etc.} \qquad \text{(iv)}$$

For $x = 0$, $d = \dfrac{1}{3\,!}$

$$9x(1-x^2)^{-\frac{5}{2}} + 15x^3(1-x^2)^{-\frac{7}{2}} = 4\,!e + 5\,!fx + 3 \,.\, 4 \,.\, 5 \,.\, 6gx^2 + \text{etc.} \qquad \text{(v)}$$

For $x = 0$, $e = 0$.

$$9(1-x^2)^{-\frac{5}{2}} + 90x^2(1-x^2)^{-\frac{7}{2}} + 105x^4(1-x^2)^{-\frac{9}{2}} = 5\,!f + 6\,!gx + \text{etc.} \quad . \quad . \qquad \text{(vi)}$$

For $x = 0$, $f = \frac{9}{5!} = \frac{1 \cdot 3}{2 \cdot 4 \cdot 5}$

Similarly, $g = 0$ and $h = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7}$ etc.

Hence the series is—

$$\sin^{-1}x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{x^9}{9} \text{ etc.}$$

The result is, of course, in radians.

If we take $x = \frac{1}{2}$, i.e. the sine of 30° or $\frac{\pi}{6}$ then

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{48} + \frac{3}{1280} + \frac{15}{336 \times 128} + \frac{105}{3456 \times 512} \text{ etc.}$$

from which $\pi = 3{\cdot}1415\ldots$

Since the sine of an angle equals the cosine of its complement it follows that

$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x = \frac{\pi}{2} - x - \frac{x^3}{2 \cdot 3} - \text{etc.}$$

NOTE: The successive differential coefficients of $(1 - x^2)^{-\frac{1}{2}}$ can be conveniently found and evaluated as follows—

Let $(1 - x^2) = z$ then for $x = 0$, $z = 1$.

If $y = (1 - x^2)^{-\frac{1}{2}}$ then $y = z^{-\frac{1}{2}}$.

Applying the rule

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

the first differentiation gives—

$$\frac{dz^{-\frac{1}{2}}}{dx} = \frac{dz^{-\frac{1}{2}}}{dz} \cdot \frac{dz}{dx} = -\frac{1}{2}z^{-\frac{3}{2}} \times -2x$$

$$= xz^{-\frac{3}{2}}$$

which for $x = 0$ is 0.

Further differentiation gives

$$\frac{d(xz^{-\frac{3}{2}})}{dx} = z^{-\frac{3}{2}} + x \times -\frac{3}{2}z^{-\frac{5}{2}} \times -2x$$
$$= z^{-\frac{3}{2}} + 3x^2z^{-\frac{5}{2}}$$

which for $x = 0$ is 1 etc.

2. $\tan^{-1} x = a + bx + cx^2 + dx^3 + \text{etc.}$ (i)

For $x = 0$, $\tan^{-1} x = 0$, $\therefore a = 0$.

By successive differentiations we obtain,

$$\frac{1}{1 + x^2} = b + 2cx + 3dx^2 + 4ex^3 + \text{etc.} \quad . \quad . \quad \text{(ii)}$$

For $x = 0$, $b = 1$

$$-2x(1 + x^2)^{-2} = 2c + 2 . 3dx + 3 . 4ex^2 + \text{etc.} \quad . \quad \text{(iii)}$$

For $x = 0$, $c = 0$

$$-2(1 + x^2)^{-2} - 4x^2(1 + x^2)^{-3} = 2 . 3d + 2 . 3 . 4ex + \text{etc.} \quad . \quad \text{(iv)}$$

For $x = 0$, $d = -\frac{1}{3}$.

Similarly $e = 0$ and $f = +\frac{1}{5}$ etc.

Hence $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$ etc.

This result could have been obtained from the fact that since (ii) is from the differentiation of (i), (i) is the integral of (ii).

Now by actual division,

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + x^8 \text{ etc.} \quad . \quad . \quad . \quad \text{(v)}$$

and this, by integration, gives,

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \text{ etc.}$$

Or the coefficients could have been obtained by comparing series (ii) and (v) term by term.

N.B. This series is convergent only within certain limits of x, namely, $+1$ and -1.

As an exercise, show that $\sinh^{-1} x = x - \frac{x^3}{3\,!} + \frac{9x^5}{5\,!}$ etc.

CHAPTER VI

VECTORS—ROTORS—OPERATORS

§1. **A meaning of j or $\sqrt{-1}$.** If OA (Fig. 9), represents a straight line of length "a" units and of positive direction from O to A, then OB of equal length but of direction precisely opposite to OA represents $-a$, i.e. $+a \times -1$.

Now OB can be regarded as the position and direction of OA after being turned, say, anti-clockwise through two right angles.

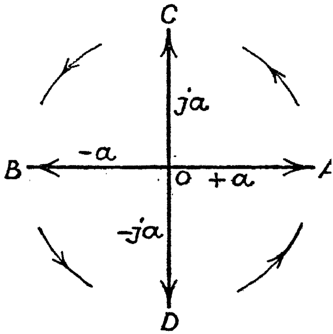

Fig. 9

Consider this rotation as occurring in two equal stages, each stage being the rotation of OA through one right angle.

Let OC represent OA after being turned through the first right angle and suppose the operation to be represented by multiplying a by j, i.e. by ja.

Then turning the straight line through the second right angle to OB it is represented by ja multiplied by j, i.e. by j^2a.

$$j^2a = -a$$

from which $$j^2 = -1$$

and $$j = \sqrt{-1}.$$

Used in this sense j is an operator which, when applied to a straight line, rotates it anti-clockwise through a right angle.

Continuing the rotation beyond OB the idea is found to be consistent.

Thus the rotation from OB to OD is represented by

$$-a \times j = -ja,$$

and from OD to OA by

$$\begin{aligned} -ja \times j &= -j^2a \\ &= -(-1)a \\ &= +a. \end{aligned}$$

Similarly it will be found consistent to consider that $-j$ applied to OA rotates it through a right angle in the clockwise direction.

This nomenclature is therefore useful in indicating direction.

Thus, while $b + a$ would be represented by two straight lines in the same direction, $b + ja$ would be shown as in Fig. 10, ja being at right angles to b.

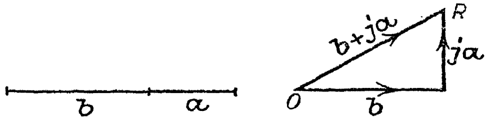

FIG. 10

OR is the resultant of b and ja and its direction is indicated by the algebraic expression $b + ja$.

OR makes angle θ with b such that $\tan\theta = \frac{a}{b}$. This angle is called the amplitude or argument of $b + ja$.

The actual length of OR is, of course, $\sqrt{a^2 + b^2}$, and this is called the modulus.

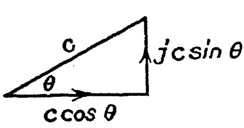

FIG. 11

In trigonometry $c \cos\theta$ and $c \sin\theta$ represent merely the lengths of the adjacent and opposite sides respectively of a right-angled triangle of which the hypotenuse is c and one acute angle θ (Fig. 11), but $c(\cos\theta + j\sin\theta)$ represents the *direction* of the hypotenuse, θ being such that

$$\tan\theta = \frac{c\sin\theta}{c\cos\theta} = \frac{\sin\theta}{\cos\theta}$$

which again is consistent and quite obvious.

The length of the hypotenuse c is also known to be equal to $\sqrt{c^2\cos^2\theta + c^2\sin^2\theta}$ since $\cos^2\theta + \sin^2\theta = 1$.

§2. **Complex Operators.** To operate on the vector $b + ja$ with $(r + jx)$ means that the vector is turned through a further angle whose tan is $\frac{x}{r}$ and its length multiplied by $\sqrt{r^2 + x^2}$.

The following analysis shows that this result follows mathematically; thus—

$$\begin{aligned}(b + ja) \times (r + jx) &= br + j^2ax + j(ar + bx)\\ &= (br - ax) + j(ar + bx).\end{aligned}$$

This result represents a vector of length

$$\sqrt{(br - ax)^2 + (ar + bx)^2} = \sqrt{b^2 + a^2} \times \sqrt{r^2 + x^2},$$

and making an angle whose tan is $\frac{ar + bx}{br - ax}$ with the zero direction. This angle is the sum of the angles whose tangents are respectively $\frac{a}{b}$ and $\frac{x}{r}$.

(*The student should satisfy himself as to this statement by using the formula* $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$.

Similarly,

(i) Operator $(r - jx)$ turns a vector through an angle

$$\tan^{-1}\left(\frac{-x}{r}\right)$$

and multiplies its length by $\sqrt{r^2 + x^2}$. Observe that $(r - jx)$ and $(r + jx)$ turn a vector in opposite directions, but both multiply the length by the same amount, namely, $\sqrt{r^2 + x^2}$.

(ii) Operator $\frac{1}{r + jx}$ divides the length of a vector by $\sqrt{r^2 + x^2}$ and turns it back through $\tan^{-1}\left(\frac{x}{r}\right)$. That is, it acts precisely opposite or inversely to operator $r + jx$.

Observe that,

$$\frac{1}{r + jx} = \frac{1 \times (r - jx)}{(r + jx) \times (r - jx)} = \frac{r - jx}{r^2 - j^2x^2} = \frac{r - jx}{r^2 + x^2}$$

Thus operator $\dfrac{1}{r + jx}$ may be considered as equivalent to operator $r - jx$ divided by $r^2 + x^2$ or

$$\sqrt{r^2 + x^2} \times \sqrt{r^2 + x^2}.$$

This is seen to be correct, for $(r - jx)$ multiplies a vector by $\sqrt{r^2 + x^2}$ so this multiplying effect is first cancelled by dividing by $\sqrt{r^2 + x^2}$, which gives the original length, which original length is then divided by $\sqrt{r^2 + x^2}$.

The expressions $(r + jx)$ and $(r - jx)$ are said to be conjugate to one another. It should be noted that the square root of their product, namely, $\sqrt{r^2 + x^2}$ is the modulus of either of them.

The reader should study closely Figs. 12 to 16.

Fig. 12 shows that $j(10 + j4) = j10 - 4$.

Fig. 13 shows vector $10 - j4$.

Fig. 14 shows that vector $- 10 + j4$ is the same as $j4 - 10$.

Fig. 15 shows that

(i) $(10 + j4) + (6 + j7) = 16 + j11$
(ii) $(10 + j4) + (6 - j7) = 16 - j3$
(iii) $(16 + j11) - (10 + j4) = 6 + j7$

Fig. 16 shows that

(i) $(10 + j4)$ operated on by $(3 + j2) = (22 + j32)$
(ii) $(10 + j4)$,, ,, ,, $(3 - j2) = (38 - j8)$
(iii) $(10 + j4)$,, ,, ,, $\dfrac{1}{3 + j2} = \dfrac{(10 + j4)(3 - j2)}{9 + 4}$

$$= \left(\frac{38}{13} - j\frac{8}{13}\right)$$

As an exercise, verify by taking suitable positive values of b and a, say $b = 10$ and $a = 6$, that vectors $(b + ja)$, $(-b + ja)$, $(-b - ja)$, $(b - ja)$ are respectively in the first, second, third and fourth quadrants, their angles, with zero direction, being respectively $\tan^{-1}\left(\dfrac{+a}{+b}\right)$, $\tan^{-1}\left(\dfrac{+a}{-b}\right)$, $\tan^{-1}\left(\dfrac{-a}{-b}\right)$, $\tan^{-1}\left(\dfrac{-a}{+b}\right)$.

§3. **Exponential Representation of a Vector.** In the form

$$b + ja \qquad \text{where } \tan\theta = \frac{a}{b}$$

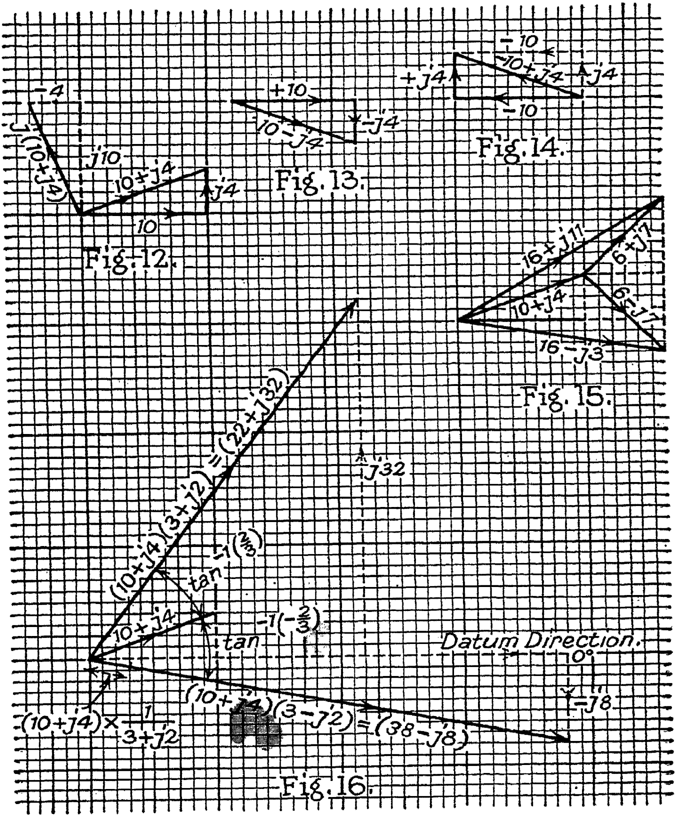

Fig. 12.

Fig. 13.

Fig. 14.

Fig. 15.

Fig. 16.

it is easily seen that since $\tan\theta = \frac{a}{b}$ then

$$\sin\theta = \frac{a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \cos\theta = \frac{b}{\sqrt{a^2+b^2}}$$

Write c for $\sqrt{a^2+b^2}$ then $a = c\sin\theta$

and $b = c\cos\theta$

$$\begin{aligned} \text{and } b + ja &= c\cos\theta + jc\sin\theta \\ &= c(\cos\theta + j\sin\theta) \end{aligned}$$

But $\cos\theta + j\sin\theta = e^{j\theta}$

and $\therefore \quad b + ja = ce^{j\theta}$

which is a still simpler representation of a vector of length c, making an angle θ with a datum direction.

Anti-clockwise measurement of θ is indicated by $+j\theta$ and clockwise by $-j\theta$.

If θ is varying, then the exponential form can represent a rotating vector. For example, if the vector makes n rev. per sec., anti-clockwise, the angle turned through in t sec. is $2\pi nt$ and the vector is represented by $ce^{j2\pi nt}$ reckoned from zero or $ce^{j(2\pi nt + \alpha)}$ if reckoned from angle α.

The following example shows how one form may be transformed into the other.

$12e^{j30°}$ represents the vector 12 units long and making an angle of 30° with the datum direction. To express it in the form $b + ja$ we may either use the form

$$c(\cos\theta + j\sin\theta)$$

from which

$$b = 12\cos 30° = 6\sqrt{3} \text{ and } a = 12\sin 30° = 6$$

or solve the equations

$$\sqrt{b^2 + a^2} = 12, \quad \frac{a}{b} = \tan 30° = \frac{1}{\sqrt{3}}$$

for b and a.

The result is $6\sqrt{3} + j6$.

It is worth noting that $6\sqrt{3}$ and 6 are the horizontal and vertical components of the vector.

Conversely, the vector $10 + j4$ (Fig. 12), expressed in exponential form is $\sqrt{116}\, e^{j(\cdot 38)}$ as $\cdot 38$ radians is $\tan^{-1}\frac{4}{10}$.

The student should practise expressing the other vectors in Figs. 12 to 16 in exponential form.

§4. **Application to Alternating Currents.** If r is the ohmic resistance of a circuit, x the reactance, I the current, and E the e.m.f. to produce current I, then the e.m.f. to overcome the ohmic resistance is rI and that to overcome the reactance xI. But these last two e.m.f.'s are at right angles as shown in Fig. 17, and E is the resultant of the two.

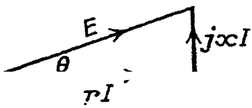

FIG. 17

Thus, $E = rI + jxI$ in direction
$$= I(r + jx)$$

The current I lags behind the voltage E by angle θ such that $\tan\theta = \frac{xI}{rI} = \frac{x}{r}$.

The magnitude of E is $I\sqrt{r^2 + x^2}$ as is evident from the sides of the right-angled triangle. The factor $\sqrt{r^2 + x^2}$ is called the impedance of the circuit.

In the relation $E = I(r + jx)$, if I is made the subject of the equation, we have,

$$I = \frac{E}{r + jx} = \frac{E(r - jx)}{r^2 - j^2x^2}$$

i.e. $$I = \frac{E}{r^2 + x^2}(r - jx)$$

This form is convenient in the summing of currents, for example in parallel circuit calculations. The ratio $\frac{-x}{r}$ indicates that I lags behind E.

For the magnitude of I we have, since the amplitude of $(r - jx)$ is $\sqrt{r^2 + x^2}$,

$$I = \frac{E}{r^2 + x^2}\sqrt{r^2 + x^2} = \frac{E}{\sqrt{r^2 + x^2}}$$

and this agrees with the previously stated magnitude of E.

The relation can be also expressed in exponential form.

Let $\quad Z = \sqrt{r^2 + x^2}$, then since

$$r + jx = Z(\cos\theta + j\sin\theta) = Ze^{j\theta} \text{ (See page 56).}$$

we may write the relation $E = I(r + jx)$ as

$$E = IZe^{j\theta}$$

which shows both the impedance and the phase relation between E and I in one equation. It will be noted that when $\theta = 0$, $e^{j\theta} = 1$ and $E = IZ$. Denoting this value of E by E_o the equation becomes

$$E = E_o e^{j\theta}$$

This exponential form is convenient when determining the product of complex expressions.

Thus if the expressions are $E_o e^{j\theta_1}$, and $I_o e^{j\theta_2}$, the product is immediately

$$E_o I_o e^{j(\theta_1 + \theta_2)} \text{ or } E_o I_o\{\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)\}$$

The student should satisfy himself that this last expression is the product of

$$E_o(\cos\theta_1 + j\sin\theta_1) \text{ and } I_o(\cos\theta_2 + j\sin\theta_2).$$

The following example illustrates the application to impedances of systems.

Consider, say, three impedances in parallel. (Fig. 18)

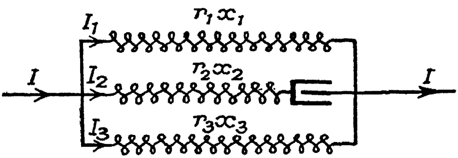

FIG. 18

Represent the resistances by r_1, r_2, r_3, the reactances by x_1, x_2, x_3 and the currents by I_1, I_2, I_3, the impressed voltage being E.

To find the single equivalent impedance Z.

The resultant current I is the vectorial sum $I_1 + I_2 + I_3$.

Now $I_1 = \frac{E}{r_1 + jx_1}$, $I_2 = \frac{E}{r_2 + jx_2}$, $I_3 = \frac{E}{r_3 + jx_3}$

and $\quad I = \frac{E}{Z}$

$$\frac{E}{Z} = \frac{E}{r_1 + jx_1} + \frac{E}{r_2 + jx_2} + \frac{E}{r_3 + jx_3}$$

or

$$\frac{1}{Z} = \frac{1}{r_1 + jx_1} + \frac{1}{r_2 + jx_2} + \frac{1}{r_3 + jx_3}$$

or multiplying numerator and denominator by $r - jx$

$$= \frac{r_1 - jx_1}{r_1^2 + x_1^2} + \frac{r_2 - jx_2}{r_2^2 + x_2^2} + \frac{r_3 - jx_3}{r_3^2 + x_3^2}$$

from which the impedance Z in operator form can be found.

Thus,

if $r_1 = 2$ and $x_1 = 3$ $\left[\text{Reactance } x = \left(2\pi fL - \frac{1}{2\pi fC}\right)\right]$

$r_2 = 4 \qquad x_2 = -1$ (Capacitance > Inductance)

$r_3 = 3 \qquad x_3 = 2$

then

$$\frac{1}{Z} = \frac{2 - j3}{4 + 9} + \frac{4 + j}{16 + 1} + \frac{3 - j2}{9 + 4}$$

$$= \left(\frac{2}{13} + \frac{4}{17} + \frac{3}{13}\right) - j\left(\frac{3}{13} - \frac{1}{17} + \frac{2}{13}\right)$$

$$= 0{\cdot}62 - j(0{\cdot}326).$$

The magnitude of $\frac{1}{Z} = \sqrt{({\cdot}62)^2 + ({\cdot}326)^2}$

$$= {\cdot}7004$$

$\therefore \qquad Z = 1{\cdot}428$ units.

Since $\qquad I = E[0{\cdot}62 - j(0{\cdot}326)]$

it follows that I lags behind E by $\tan^{-1}\left(\frac{\cdot 326}{\cdot 62}\right) = \tan^{-1}(\cdot 526)$, which, by tables, is 28° approx.

Students of electrical engineering know that the power factor of the system is given by the cosine of the angle of phase difference. In this case it is $\cos 28° = \cdot 8829$ or approximately 88 per cent.

If $E = 100$ volts, then,

$$I_1 = \frac{100}{2 + j3} = \frac{100}{13}(2 - j3) \text{ which lags by } \tan^{-1}\frac{3}{2} \simeq 56\tfrac{1}{2}°$$

the magnitude being $\frac{100}{13}\sqrt{4 + 9} = 27\cdot 73$ amp.

$$I_2 = \frac{100}{4 - j} = \frac{100}{17}(4 + j) \text{ which leads by } \tan^{-1}\frac{1}{4} \simeq 14°$$

the magnitude being $\frac{100}{17}\sqrt{16 + 1} = 24\cdot 25$ amp.

$$I_3 = \frac{100}{3 + j2} = \frac{100}{13}(3 - j2), \text{ which lags by } \tan^{-1}\frac{2}{3} \simeq 34°$$

the magnitude being $\frac{100}{13}\sqrt{13} = 27\cdot 73$ amp.

and $I = \frac{100}{1\cdot 428} = 70\cdot 04$ amp., and lags by 28° approx.

As a check we can represent the currents in exponential form, thus

$$\begin{aligned}
I_1 &= 27\cdot 73e^{-j56\frac{1}{2}°} = 27\cdot 73(\cos 56\tfrac{1}{2}° - j \sin 56\tfrac{1}{2}°) \\
&\qquad\qquad = 15\cdot 31 - j23\cdot 1 \\
I_2 &= 24\cdot 25e^{j14°} = 24\cdot 25(\cos 14° + j \sin 14)° \\
&\qquad\qquad = 23\cdot 62 + j5\cdot 89 \\
I_3 &= 27\cdot 73e^{-j34°} = 27\cdot 73(\cos 34° - j \sin 34)° \\
&\qquad\qquad = \underline{22\cdot 99 - j15\cdot 5} \\
\therefore \qquad I_1 + I_2 + I_3 &= I = 61\cdot 92 - j32\cdot 7
\end{aligned}$$

which is approximately $100[0\cdot 62 - j(0\cdot 326)]$

The equivalent impedance of these three impedances in series is simply,

$$\{(r_1 + r_2 + r_3) + j(x_1 + x_2 + x_3)\}$$

the current lagging by

$$\tan^{-1}\frac{x_1 + x_2 + x_3}{r_1 + r_2 + r_3}$$

and its magnitude is

$$\sqrt{(r_1 + r_2 + r_3)^2 + (x_1 + x_2 + x_3)^2}$$

The student should work this out for himself and refer to textbooks on electrical engineering for examples to solve by the foregoing method. Suitable exercises will be found in *Classified Examples in Electrical Engineering*, Vol. II (S. G. Monk), Sir I. Pitman and Sons, Ltd.

The following further example illustrates the method of finding the equivalent impedance of an impedance (Z_a) in series with two impedances (Z_b and Z_c) in parallel.

The data are as follows:

Frequency 50 cycles per second.

Z_a: Resistance	2 ohms,	inductance	·01 henries
Z_b	1 ohm,	,,	·02 ,,
Z_c	3 ohms,	capacity	5000 microfarads.

The reactances of Z_a, Z_b, Z_c are, respectively:

$$2\pi \times 50 \times \cdot 01 = \pi,\ 2\pi \times 50 \times \cdot 02 = 2\pi, \text{ and}$$

$$-\frac{1}{2\pi \times 50 \times \cdot 005} = -\frac{2}{\pi}$$

If z is the joint impedance of Z_b and Z_c in parallel, then,

$$\frac{1}{z} = \frac{1}{1 + j2\pi} + \frac{1}{3 - j\frac{2}{\pi}}$$

$$= \frac{3 - j\frac{2}{\pi} + 1 + j2\pi}{\left(1 + j2\pi\right)\left(3 - j\frac{2}{\pi}\right)}$$

$$= \frac{4 + j5{\cdot}65}{7 + j18{\cdot}2}$$

$$\therefore\ z = \frac{7 + j18{\cdot}2}{4 + j5{\cdot}65} = \frac{130{\cdot}8 + j33{\cdot}25}{47{\cdot}92}$$

$$\simeq 2{\cdot}73 + j{\cdot}7.$$

If Z is the equivalent impedance of the system, then,

$$\begin{aligned} Z &= Z_a + z \\ &= (2 + j\pi) + (2{\cdot}73 + j{\cdot}7) \\ &= 4{\cdot}73 + j3{\cdot}84 \end{aligned}$$

The magnitude is $\sqrt{(4{\cdot}73)^2 + (3{\cdot}84)^2}$

$\simeq 6{\cdot}1$ units.

The angle of lag is $\tan^{-1}\dfrac{3{\cdot}84}{4{\cdot}73} \simeq 39°$.

If the impressed volts $= 100$

then $$I = \frac{E}{Z} = \frac{100}{4{\cdot}73 + j3{\cdot}84}$$

$$= \frac{100}{37{\cdot}1}(4{\cdot}73 - j3{\cdot}84)$$

$$= 12{\cdot}75 - j10{\cdot}35 \quad . \quad . \quad . \quad \text{(i)}$$

The magnitude $= \sqrt{(12{\cdot}75)^2 + (10{\cdot}35)^2} = 16{\cdot}4$ amp. (ii)

lagging at $\tan^{-1}\dfrac{10{\cdot}35}{12{\cdot}75} = 39°$.

Now if I_b and I_c are the currents in the parallel branches b and c respectively, then

$$I_b + I_c = I \text{ (in the vector sense)}$$

But $$\frac{I_b}{I_c} \quad \frac{3 - j\frac{2}{\pi}}{1 + j2\pi} \text{ (inversely as their impedances)}$$

$$I_b + I_c \quad \frac{3 - j\frac{2}{\pi}}{4 + j\left(2\pi - \frac{2}{\pi}\right)} = \frac{8\cdot4 - j19\cdot5}{48}$$

i.e. $$I_b = \frac{8\cdot4 - j19\cdot5}{48} I \qquad \text{. (iii)}$$

Similarly, $$I_c = \frac{39\cdot5 + j19\cdot5}{48} I \qquad \text{. (iv)}$$

Hence $$I_b = \frac{8\cdot4 - j19\cdot5}{48}\,(12\cdot75 - j10\cdot35) \qquad \text{from (i)}$$
$$= -\,1\cdot97 - j7 \text{ (direction)}$$
$$= \sqrt{(1\cdot97)^2 + 7^2} \text{ (magnitude)}$$
$$= 7\cdot27 \text{ amp., leading by } \tan^{-1}\left(\frac{-\,7}{-\,1\cdot97}\right)$$

which is an angle in the third quadrant, namely 254°, or lagging by 106°

and $$I_c = \frac{39\cdot5 + j19\cdot5}{48}\,(12\cdot75 - j10\cdot35) \text{ from (iv and i)}$$
$$= 14\cdot7 - j3\cdot32 \text{ (direction)}$$
$$= \sqrt{(14\cdot7)^2 + (3\cdot32)^2} \text{ (magnitude)}$$
$$= 15\cdot1 \text{ amps. approx., lagging at } 13^\circ \text{ nearly.}$$

Observe that $(-\,1\cdot97 - j7) + (14\cdot7 - j3\cdot32)$
$$= 12\cdot73 - j10\cdot32 = I \text{ nearly,}$$

which checks the computation.

The student should draw a vector diagram of these currents, making the direction of the impressed e.m.f. the zero.

It will be seen that I_c leads I by $(39^\circ - 13^\circ) = 26^\circ$,

I_b lags behind I by $(106° - 39°) = 67°$ and that $I = I_b + I_c$ vectorially.

§5. **Symbolic Representation of Forces.** The sides of one of the triangles in Fig. 15, page 55, say the triangle with sides $(10 + j4)$, $(6 - j7)$, $(16 - j3)$, may be taken to represent three forces, in which case it is seen that force $(16 - j3)$ is the resultant of the other two.

Observe that $(16 - j3)$ is the sum of $(10 + j4)$ and $(6 - j7)$.

For equilibrium the force $(16 - j3)$ must be reversed. This reversed force is represented by $-16 + j3$.

Thus for equilibrium we have

$$(10 + j4) + (6 - j7) + (-16 + j3) = 0$$

and this suggests at once a ready method of solving problems in statics. The magnitudes of the forces are respectively $\sqrt{10^2 + 4^2}$, $\sqrt{6^2 + 7^2}$, $\sqrt{16^2 + 3^2}$.

The same relation applies to the polygon of forces.

The resultant of forces—

$$\begin{array}{lr} & 8 + j5 \\ & -3 + j6 \\ & 12 - j9 \\ & \underline{-7 - j4} \\ \text{is the sum} & \underline{\underline{10 - j2}} \end{array}$$

and the force to produce equilibrium is therefore $-10 + j2$ in direction, and $\sqrt{104}$ in magnitude.

As an exercise, find the resultant of the following coplanar forces acting at a point, viz.:

10 lb. at 0°, 15 lb. at 60°, 24 lb. at 150°, 20 lb. at 220°, 30 lb. at 300°

These forces are easily expressed algebraically through the form $c(\cos\theta + j\sin\theta)$, e.g.

20 lb. at 220° $= 20(\cos 220° + j\sin 220°) = -15{\cdot}32 - j12{\cdot}86$.

The sum is $(-3{\cdot}6 - j13{\cdot}85)$

magnitude $= \sqrt{3{\cdot}6^2 + 13{\cdot}85^2} = 14{\cdot}3$ lb.

$$\theta = \tan^{-1}\frac{-13{\cdot}85}{-3{\cdot}6} = 255° \text{ (3rd quadrant)}$$

§6. **The Roots of $+1$ and -1.**

Since $\cos 2\pi = +1$ and $\sin 2\pi = 0$ it follows that

$$\cos 2\pi \pm j \sin 2\pi = +1$$

and $\therefore$ $$(\cos 2\pi \pm j \sin 2\pi)^{\frac{1}{n}} = (+1)^{\frac{1}{n}}$$

But by De Moivre's theorem,

$$(\cos 2\pi \pm j \sin 2\pi)^{\frac{1}{n}} = \cos \frac{2\pi}{n} \pm j \sin \frac{2\pi}{n}$$

$$\therefore \qquad \cos \frac{2\pi}{n} \pm j \sin \frac{2\pi}{n} = (+1)^{\frac{1}{n}}$$

This relation enables us to calculate the complex roots of $+1$.

E.g. for the cube roots, $\frac{1}{n} = \frac{1}{3}$, and $\cos \frac{2\pi}{3} = -\frac{1}{2}$ and $\sin \frac{2\pi}{3} = +\frac{\sqrt{3}}{2}$

Hence two of the cube roots of $(+1) = \left(-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}\right)$

If the student cubes this expression he will obtain $+1$. There are three cube roots of $+1$, namely,

$$+1, \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) \text{ and } \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

Similarly, since $\cos \pi = -1$ and $\sin \pi = 0$

it follows that $\cos \pi \pm j \sin \pi = -1$

and $$(\cos \pi \pm j \sin \pi)^{\frac{1}{n}} = (-1)^{\frac{1}{n}}$$

and $\therefore$ $$\cos \frac{\pi}{n} \pm j \sin \frac{\pi}{n} = (-1)^{\frac{1}{n}}$$

which relation gives the complex roots of (-1).

For example, for the cube roots of (-1), $\frac{1}{n} = \frac{1}{3}$,

$$\cos \frac{\pi}{3} = +\frac{1}{2}, \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \text{ and } \therefore \sqrt[3]{-1} = \left(\frac{1}{2} \pm j\frac{\sqrt{3}}{2}\right)$$

For the square root we have, $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$

and $\therefore \quad \sqrt{-1} = 0 \pm j$

$$= \pm j$$

which is consistent with the fundamental definition of j.

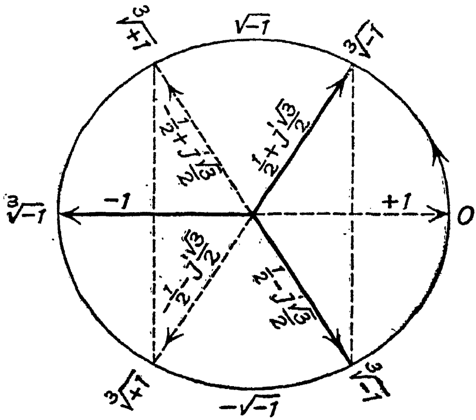

FIG. 19

The diagram known as Argand's diagram illustrates the relation graphically. (Fig. 19.)

EXERCISES

1. Verify that the third cube-root of unity is the square of the second; also that if vector $\left(-\frac{1}{2} + j\frac{\sqrt{3}}{\cancel{3}\,2}\right)$ is operated on by $\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$ the resultant vector is $-\frac{1}{2} - j\frac{\sqrt{3}}{2}$ the length of which is 1.

2. Express the complex cube roots of $+1$ and -1 in exponential form.

§7. Quadratic Equations with Imaginary Roots—Graphical Representation.*

EXAMPLE 1

$$x^2 + 4 = 0$$

The roots are $\qquad x = \pm j2$

In the graphical representation the multiplier j turns the axis of x through a right angle and therefore we can consider real and imaginary values of x being graphed respectively in planes at right angles.

Calculate y when $y = x^2 + 4$, for $x = 0, \pm j, \pm j2, \pm j3$, etc., and plot y against x.

$-j3$	$-j2$	$-j$	$= x =$	0	$+j$	$+j2$	$+j3$
-5	0	3	$= y = (x^2 + 4) =$	4	3	0	-5

The graph cuts the axis of x at $\pm j2$, which values are the roots of the equation $x^2 + 4 = 0$.

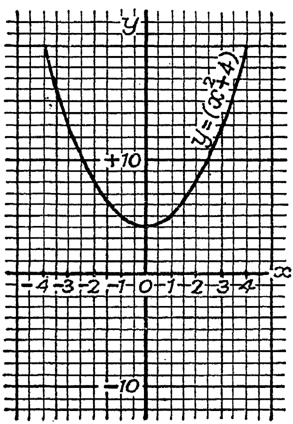

FIG. 20

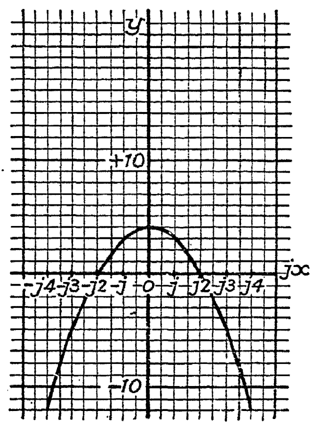

FIG. 21

Figs. 20 and 21 are in planes at right angles to one another.

EXAMPLE 2

$$x^2 - 2x + 5 = 0$$
$$x^2 - 2x + 5 = (x - 1)^2 + 4$$

By putting Z for $(x - 1)$ the function $y = x^2 - 2x + 5$ may be written $y = Z^2 + 4$.

Plot the graph of $y = Z^2 + 4$ for $Z = 0, \pm j, \pm j2$, etc. The graph cuts the axis of Z at $\pm j2$.

The roots are $Z = \pm j2$

i.e. $x - 1 = \pm j2$

from which $x = 1 \pm j2$

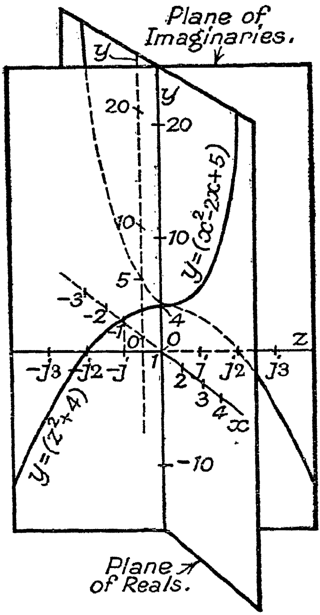

FIG. 22

Fig. 22 shows the two graphs in perspective, the imaginary values of Z being in full view (plane of the paper) and the real values of x being in a plane at right angles to the paper. The axis of Z passes at right angles through the axis of x at $x = 1$. (For $Z = 0$, $x = 1$.)

The student should substitute $1 \pm j2$ for x in Example 2 and verify the result.

§8. **Further Deduction involving the Imaginary $\sqrt{-1}$.**

Since $$e^{jx} = \cos x + j \sin x \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

then for $$x = \frac{\pi}{2}\left(\text{more generally for } (4n+1)\frac{\pi}{2}\right)^{\dagger}$$

$$e^{j\frac{\pi}{2}} = j\left(\text{since } \cos\frac{\pi}{2} = 0, \text{ and } \sin\frac{\pi}{2} = 1\right) \qquad \text{(ii)}$$

and raising to the power j

$$e^{-\frac{\pi}{2}} = j^j \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(iii)}$$

Again, for $x = \pi$ we have from (i),

$$e^{j\pi} = -1$$

from which $$\pi = \frac{\text{logh}(-1)}{j} = \frac{\text{logh}(-1)}{\sqrt{-1}} \quad . \quad . \qquad \text{(iv)}$$

which is a new aspect of π.

As an exercise, show that $j^j = 0{\cdot}2078$.

† As j^j has many values the first value (iii) is called the Principal Value.

CHAPTER VII

SUMMARY AND APPLICATIONS

§1. Exponential Relations.

(i) $e^{\pm x} = 1 \pm x + \frac{x^2}{2!} \pm \frac{x^3}{3!} + \frac{x^4}{4!}$ etc.

(ii) $\cosh x = \frac{e^x + e^{-x}}{2}$

(iii) $\sinh x = \frac{e^x - e^{-x}}{2}$

(iv) $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

(v) $\cosh x \pm \sinh x = e^{\pm x}$

(vi) $\cosh^2 x - \sinh^2 x = 1$

(vii) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(viii) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

(ix) $\cosh 2x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$

(x) $\sinh 2x = 2\cosh x \sinh x$

(xi) $\cosh^{-1}\frac{x}{a} = \text{logh}\,\frac{x \pm \sqrt{x^2 - a^2}}{a}$

(xii) $\sinh^{-1}\frac{x}{a} = \text{logh}\,\frac{x +}{a}$

(xiii) $\tanh^{-1}\frac{x}{a} = \frac{1}{2}\,\text{logh}\,\frac{a + x}{a - x}$

(xiv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} +$ etc.

(xv) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} +$ etc.

(xvi) $e^{\pm jx} = \cos x \pm j \sin x$

$$\left.\begin{array}{ll}\text{(xvii)} & \cos x = \dfrac{e^{jx} + e^{-jx}}{2} = \cosh jx \\ \text{(xviii)} & \sin x = \dfrac{e^{jx} - e^{-jx}}{2j} = \dfrac{\sinh jx}{j}\end{array}\right\}\text{Euler's expressions}$$

(xix) $\cos jx = \dfrac{e^{x} + e^{-x}}{2} = \cosh x$

(xx) $\sin jx = j\dfrac{e^{x} - e^{-x}}{2} = j \sinh x$

(xxi) $\cosh(a \pm jb) = \cosh a \cos b \pm j \sinh a \sin b$

(xxii) $\sinh(a \pm jb) = \sinh a \cos b \pm j \cosh a \sin b$

(xxiii) $(\cos x \pm j \sin x)^n = \cos nx \pm j \sin nx$ (De Moivre)
$= e^{\pm jnx}$

(xxiv) $e^{j\pi} = -1$

(xxv) $e^{-\frac{\pi}{2}} = j^j$

(xxvi) $\sin^{-1}x = x + \dfrac{1 \,.\, x^3}{2 \,.\, 3} + \dfrac{1 \,.\, 3 \,.\, x^5}{2 \,.\, 4 \,.\, 5} + \dfrac{1 \,.\, 3 \,.\, 5x^7}{2 \,.\, 4 \,.\, 6 \,.\, 7} + \text{etc.}$

(xxvii) $\tan^{-1}x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \text{etc.}$

(xxviii) $\operatorname{logh}(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4}\ \text{etc.}$

Differential Coefficients and Integrals.

(xxix) $\dfrac{d}{dx}(e^{kx}) = ke^{kx}, \quad \displaystyle\int e^{kx}\, dx = \frac{e^{kx}}{k}$

(xxx) $\dfrac{d}{dx}(\operatorname{logh} x) = \dfrac{1}{x}, \quad \displaystyle\int \frac{dx}{x} = \operatorname{logh} x$

(xxxi) $\dfrac{d}{dx}(e^{k-ax}) = -ae^{k-ax}, \quad \displaystyle\int e^{k-ax}\, dx = -\frac{e^{k-ax}}{a}$

(xxxii) $\dfrac{d}{dx}(\text{logh } (K \pm x)) = \pm \dfrac{1}{K \pm x}$,

$$\int \frac{dx}{K \pm x} = \pm \text{ logh } (K \pm x)$$

(xxxiii) $\dfrac{d}{dx}(a^x) = \text{logh } a \,.\, a^x, \quad \displaystyle\int a^x dx = \frac{a^x}{\text{logh } a}$

(xxxiv) $\dfrac{d}{dx}(\cosh x) = \sinh x, \quad \displaystyle\int \sinh x \, dx = \cosh x$

(xxxv) $\dfrac{d}{dx}(\sinh x) = \cosh x, \quad \displaystyle\int \cosh x \, dx = \sinh x$

(xxxvi) $\dfrac{d}{dx}(\tanh x) = \text{sech}^2 x, \quad \displaystyle\int \text{sech}^2 x \, dx = \tanh x$

(xxxvii) $\dfrac{d}{dx}\left(\cosh^{-1}\dfrac{x}{a}\right) = \dfrac{1}{\sqrt{x^2 - a^2}}, \quad \displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a}$

(xxxviii) $\dfrac{d}{dx}\left(\sinh^{-1}\dfrac{x}{a}\right) = \dfrac{1}{\sqrt{x^2 + a^2}}, \displaystyle\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a}$

(xxxix) $\dfrac{d}{dx}\left(\tanh^{-1}\dfrac{x}{a}\right) = \dfrac{a}{a^2 - x^2}, \quad \displaystyle\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a}$

(xl) $\displaystyle\int \tanh x \, dx = \text{logh } \cosh x$

(xli) $\displaystyle\int \coth x \, dx = \text{logh } \sinh x$

(xlii) $\displaystyle\int \frac{dx}{\sinh x} = \text{logh } \tanh \frac{x}{2}$

(xliii) $\displaystyle\int \frac{dx}{\cosh x} = 2 \tan^{-1} e^x$

(The constant of integration is to be added to each integral.)

§2. Applications to Integration.

I. $$\int e^{jbx}\,dx = \frac{1}{jb}\,e^{jbx} = \frac{1}{jb}(\cos bx + j\sin bx)$$

i.e. $$\int(\cos bx + j\sin bx)\,dx = \frac{1}{jb}(\cos bx + j\sin bx)$$

II. (*a*) $$\int \sin^2 x\,dx = \int\left(\frac{e^{jx}-e^{-jx}}{2j}\right)^2 dx$$

$$= -\frac{1}{4}\int(e^{2jx}+e^{-2jx}-2)dx$$

$$= -\frac{1}{4}\left[\frac{e^{2jx}}{2j}-\frac{e^{-2jx}}{2j}-2x\right]$$

$$= -\frac{1}{4}\sin 2x + \frac{x}{2}$$

(*b*) $$\int \cos^3 x\,dx = \int\left(\frac{e^{jx}+e^{-jx}}{2}\right)^3 dx$$

$$= \frac{1}{8}\int(e^{3jx}+3e^{2jx}e^{-jx}+3e^{jx}e^{-2jx}+e^{-3jx})dx$$

$$= \frac{1}{8}\left[\frac{e^{3jx}}{3j}+\frac{3e^{jx}}{j}-\frac{3e^{-jx}}{j}-\frac{e^{-3jx}}{3j}\right]$$

$$= \frac{1}{8}\left[\frac{e^{3jx}-e^{-3jx}}{3j}+3\frac{e^{jx}-e^{-jx}}{j}\right]$$

$$= \frac{1}{4}\left[\frac{\sin 3x}{3}+3\sin x\right]$$

or $$= \sin x - \frac{\sin^3 x}{3}$$

III. $$\int e^{(a+jb)x}dx = \frac{e^{(a+jb)x}}{a+jb} \quad \frac{a-jb}{a^2+b^2}e^{(a+jb)x}$$

i.e. $$\int e^{ax}(\cos bx + j\sin bx)\,dx,$$

or $$\int e^{ax}\cos bx\,dx + \int e^{ax}j\sin bx\,dx$$

$$= \frac{a - jb}{a^2 + b^2}e^{ax}(\cos bx + j\sin bx)$$

$$= \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx)$$

$$+ \frac{je^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx)$$

Equating real to real, and then imaginary to imaginary, we have,

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx)$$

and $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx)$$

As an exercise, find the average value of

$$e^{-\frac{x}{10}}\cos 3x \text{ from } x = 2\pi \text{ to } \frac{5}{2}\pi$$

IV. $$\int e^{x}\cos x\sin x\,dx = \frac{1}{2}\int e^{x}\sin 2x\,dx$$

$$= \frac{1}{2}\int e^{x} \cdot \frac{e^{2jx} - e^{-2jx}}{2j}dx$$

$$= \frac{1}{4j}\int (e^{(1 + 2j)x} - e^{(1 - 2j)x})dx$$

$$= \frac{1}{4j}\left[\frac{e^{(1 + 2j)x}}{1 + 2j} - \frac{e^{(1 - 2j)x}}{1 - 2j}\right]$$

$$= \frac{e^{x}}{20j}\left[(1 - 2j)e^{2jx} - (1 + 2j)e^{-2jx}\right]$$

$$e^{x}\left[\frac{e^{2jx} - e^{-2jx}}{j} - 2j\frac{e^{2jx} + e^{-2jx}}{j}\right]$$

$$= \frac{e^x}{20}(2 \sin 2x - 4 \cos 2x)$$

$$= \frac{e^x}{10}(\sin 2x - 2 \cos 2x)$$

7. $\int \cosh x \cos x \, dx$ and $\int \sinh x \sin x \, dx$.

By (xxi) (Summary).

$$\cosh x \cos x + j \sinh x \sin x = \cosh(1 + j)x$$

$$\therefore \int \cosh x \cos x \, dx + j \int \sinh x \sin x \, dx$$

$$= \int \cosh(1 + j)x \, dx$$

$$= \frac{\sinh(1 + j)x}{1 + j}$$

$$= \frac{(1 - j) \sinh (1 + j)x}{2}$$

$$= \frac{1}{2}\{(1 - j) (\sinh x \cos x + j \cosh x \sin x)\}$$

$$= \frac{1}{2}\{(\sinh x \cos x + \cosh x \sin x)$$
$$+ j(\cosh x \sin x - \sinh x \cos x)\}$$

Equating real to real and then imaginary to imaginary, we have,

$$\int \cosh x \cos x \, dx = \frac{1}{2}(\sinh x \cos x + \cosh x \sin x)$$

and $$\int \sinh x \sin x \, dx = \frac{1}{2}(\cosh x \sin x - \sinh x \cos x)$$

As an exercise, beginning with relation (xxii) (Summary), show that $\int \sinh x \cos x \, dx = \frac{1}{2}(\cosh x \cos x + \sinh x \sin x)$

and $\int \cosh x \sin x \, dx = \frac{1}{2}(\sinh x \sin x - \cosh x \cos x)$.

§ 3. **Logarithmic Differentiation.** The differentiation of a function consisting of the product (or quotient) of several functions yields readily to logarithmic treatment.

Let $y = uv/w$ (all functions of x) . . . . (i)

Then since

$$\text{logh}\, y = \text{logh}\, u + \text{logh}\, v - \text{logh}\, w \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

and $\dfrac{dY}{dx} = \dfrac{dY}{dZ} \cdot \dfrac{dZ}{dx}$ where Z represents in turn each of these functions of x, and $Y = \text{logh}\, Z$, we have by differentiating equation (ii)

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} - \frac{1}{w}\frac{dw}{dx} \quad . \qquad \text{(iii)}$$

r $$\frac{dy}{dx} = y\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} - \frac{1}{w}\frac{dw}{dx}\right) \qquad \text{(iv)}$$

EXAMPLE 1.

If $y = x^n e^{ax} \sin bx$
then by (iv)

$$\frac{dy}{dx} = x^n e^{ax} \sin bx\left(\frac{nx^{n-1}}{x^n} + \frac{ae^{ax}}{e^{ax}} + \frac{b \cos bx}{\sin bx}\right)$$

$$= nx^{n-1}e^{ax} \sin bx + ae^{ax}\, x^n \sin bx + x^n\, e^{ax}\, b \cos bx$$

The student should also work this example through the logarithmic equation, beginning,

$$\text{logh}\, y = n\, \text{logh}\, x + ax + \text{logh} \sin bx$$

This method is adopted in the next example.

EXAMPLE 2.

If $y = \dfrac{x^n\, e^{ax}}{\tan bx}$

then $\text{logh}\, y = n\, \text{logh}\, x + ax - \text{logh} \tan bx$, and differentiating,

$$\frac{1}{y}\frac{dy}{dx} = \frac{n}{x} + a - \frac{b \sec^2 bx}{\tan bx}$$

$$\therefore \frac{dy}{dx} = y\left(\frac{n}{x} + a - \frac{b \sec^2 bx}{\tan bx}\right)$$

$$= \frac{x^n e^{ax}}{\tan bx}\left(\frac{n}{x} + a - \frac{b \sec^2 bx}{\tan bx}\right)$$

EXAMPLE 3.

Find the fractional change in $e^{-\frac{x}{10}} \sin x$ when x changes by $\cdot 01$ from $\frac{\pi}{4}$.

Let $y = e^{-\frac{x}{10}} \sin x$

then $\operatorname{logh} y = -\frac{x}{10} + \operatorname{logh} \sin x$

and $\frac{1}{y}\frac{dy}{dx} = -\frac{1}{10} + \frac{\cos x}{\sin x}$

Again, since $\frac{\delta y}{\delta x} \doteqdot \frac{dy}{dx}$

we have by substituting $\frac{\delta y}{\delta x}$ for $\frac{dy}{dx}$ and transposing,

$$\frac{\delta y}{y} \doteqdot \left(-\frac{1}{10} + \cot x\right) \delta x$$

$$\doteqdot \left(-\frac{1}{10} + 1\right) \times \cdot 01$$

$$\doteqdot \cdot 009$$

EXERCISES

1. Differentiate (i) $x^3 e^{2x} \sin \frac{1}{2}x$. (ii) $e^{-2x}\sqrt{x} \cos x$. (iii) $e^{x^2} \sin^2 x$. (iv) e^{2x}/x^2.

2. Find the fractional change in $e^{-\frac{1}{2}x} \cos 2x$ when x changes from $\frac{1}{6}\pi$ by $\cdot 01$.

EXERCISES

Show that—

1. $\tanh jx = j \tan x$.
2. $\tan jx = j \tanh x$.
3. $\cosh(x + j2n\pi) = \cosh x$.
4. $\sinh(x + j2n\pi) = \sinh x$.
5. $\cosh\left(x + j\frac{\pi}{2}\right) = j \sinh x = \sin jx$.
6. $\sinh\left(x + j\frac{\pi}{2}\right) = j \cosh x = j \cos jx$.
7. $\tanh\left(x + j\frac{\pi}{2}\right) = \coth x$.
8. $e^{-j\frac{\pi}{2}} = -j$, and $\operatorname{logh}(-j) = -j\frac{\pi}{2}$.
9. $\operatorname{logh}(-1) = \pm j\pi$.
10. $$\cosh u = \frac{1 + \tanh^2 \frac{u}{2}}{1 - \tanh^2 \frac{u}{2}}$$
11. $$\sinh u = \frac{2 \tanh \frac{u}{2}}{1 - \tanh^2 \frac{u}{2}}.$$
12. $$e^u = \frac{1 + \tanh \frac{u}{2}}{1 - \tanh \frac{u}{2}}.$$
13. If $\tan\frac{\theta}{2} = \tanh \frac{u}{2}$, then $\sec \theta = \cosh u$,

and $$e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right).$$

14. If $\tan \frac{\theta}{2} = \tanh \frac{u}{2}$, and $\theta = 50°$, then $u = 1{\cdot}01$.
15. $\int_0^{[illegible]} e^x \cos x \, dx = 1{\cdot}9$.

16. $\int_0^{\frac{1}{2}\pi} e^x \sin x \, dx = 2{\cdot}9.$

17. $\int_0^{\frac{1}{2}\pi} e^x \cos x \sin x \, dx = 1{\cdot}16.$

18. $\int_0^{\frac{1}{2}\pi} \cosh x \cos x \, dx = \frac{1}{2} \cosh \frac{1}{2}\pi = 1{\cdot}25.$

19. $\int_0^{\pi} \sinh x \sin x \, dx = \frac{1}{2} \sinh \pi = 5{\cdot}78.$

20. $\int_0^{\pi} \cosh x \sin x \, dx = \frac{1}{2} (\cosh \pi + 1) = 6{\cdot}3.$

ANSWERS TO EXERCISES

Chapter I. *Exercise,* Page 6

1. $e^{\cdot 693}$, $e^{2\cdot 3026}$, $e^{-1\cdot 204}$, $e^{\cdot 346}$.
2. 20·086, 1·39, 4·1, 1·648, 4·81, ·208.
3. – 0·695.

Exercise, Page 8

3. 1·718. (4) $1{\cdot}0986k$, $k \operatorname{logh} \dfrac{x_2}{x_1}$.

Chapter II

Page 13 36·16 millions — actual was 36·07 millions.
,, 14 £199 (approx.), 1·3 years.
,, 16 59° (approx.).
,, 20 4·8 lb., 111 lb. (approx.).
,, 21 24·5 in. (approx.).
,, 22 (1) 56·49 sec., ·6475 tons.
(2) $v = \dfrac{V_o}{1 + V_oKt}$, $s = \dfrac{1}{K} \operatorname{logh}(1 + V_oKt)$.
,, 24 (2) $\cdot 04(e^{\cdot 5\theta_2} - e^{\cdot 5\theta_1})$, ·152, 1·09.

Chapter III. Page 32

1. $(x^2 - 1)^{-\frac{1}{2}} - x(x^2 - 1)^{-\frac{3}{2}}$, $(2x^2 + 1)(x^2 - 1)^{-\frac{5}{2}}$.
$(x^2 + 1)^{-\frac{1}{2}}$, $-x(x^2 + 1)^{-\frac{3}{2}}$, $(2x^2 - 1)(x^2 + 1)^{-\frac{5}{2}}$.

3. (i) $\sinh^{-1} \dfrac{x + 3}{2}$, (ii) $\cosh^{-1} \dfrac{x + 3}{2}$, (iii) $\sin^{-1} \dfrac{x + 3}{2}$.

4. (i) $\sinh^{-1} \dfrac{x + 3}{2}$, (ii) $\sinh^{-1} \dfrac{x + 2}{3}$, (iii) $\sinh^{-1} \dfrac{x - 2}{3}$,

(iv) $\sinh^{-1}\dfrac{x+2}{\sqrt{5}}$, (v) $\cosh^{-1}\dfrac{x-2}{\sqrt{5}}$, (vi) $\cosh^{-1}(2x+5)$.

Page 35

(1) π. (2) 2·64. (3) $\dfrac{x}{2}\sqrt{x^2-4}-2\cosh^{-1}\dfrac{x}{2}$.

(4) $\dfrac{x}{2}\sqrt{x^2+25}+12\tfrac{1}{2}\sinh^{-1}\dfrac{x}{5}$. (5) 64·37.

Page 38

(3) 20·036. (4) $3\sinh 3t$, $9\cosh 3t$.

Chapter V. Page 46

$\cos 2x = 2\cos^2 x - 1$, $\sin 2x = 2\sin x\cos x$,
$\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$,
$\sin 4x = 4\cos x\sin x - 8\cos x\sin^3 x$.

Chapter VI. Page 57

$$
\begin{aligned}
10+j4 &= \sqrt{116}(\cos 22^\circ + j\sin 22^\circ) &&= \sqrt{116}e^{j\cdot 384}\\
-4+j10 &= \sqrt{116}(\cos 112^\circ + j\sin 112^\circ) &&= \sqrt{116}e^{j1\cdot 95}\\
10-j4 &= \sqrt{116}(\cos 22^\circ - j\sin 22^\circ) &&= \sqrt{116}e^{-j\cdot 384}\\
-10+j4 &= \sqrt{116}(\cos 158^\circ + j\sin 158^\circ) &&= \sqrt{116}e^{j2\cdot 76}\\
6+j7 &= \sqrt{85}(\cos 49^\circ + j\sin 49^\circ) &&= \sqrt{85}e^{j\cdot 86}\\
6-j7 &= \sqrt{85}(\cos 49^\circ - j\sin 49^\circ) &&= \sqrt{85}e^{-j\cdot 86}\\
16+j11 &= \sqrt{377}(\cos 44^\circ + j\sin 44^\circ) &&= \sqrt{377}e^{j\cdot 76}\\
16-j3 &= \sqrt{265}(\cos 11^\circ - j\sin 11^\circ) &&= \sqrt{265}e^{-j\cdot 19}\\
3+j2 &= \sqrt{13}(\cos 42^\circ + j\sin 42^\circ) &&= \sqrt{13}e^{j\cdot 72}\\
3-j2 &= \sqrt{13}(\cos 42^\circ - j\sin 42^\circ) &&= \sqrt{13}e^{-j\cdot 72}\\
22+j32 &= 38\cdot 7(\cos 56^\circ + j\sin 56^\circ) &&= 38\cdot 7e^{j\cdot 98}\\
22-j8 &= 23\cdot 2(\cos 20^\circ - j\sin 20^\circ) &&= 23\cdot 2e^{-j\cdot 349}
\end{aligned}
$$

Page 66 (2) $e^{j\frac{\pi}{3}}$, $e^{-j\frac{\pi}{3}}$.

Chapter VII. Page 74

$-$ 0·093 (approx.).

Page 77

1. (i) $3x^2e^{2x}\sin\tfrac{1}{2}x + 2e^{2x}x^3\sin\tfrac{1}{2}x + \tfrac{1}{2}x^3e^{2x}\cos\tfrac{1}{2}x$

   (ii) $-2e^{-2x}\sqrt{x}\cos x + \tfrac{1}{2}\dfrac{e^{-2x}\cos x}{\sqrt{x}} - e^{-2x}\sqrt{x}\sin x$

   (iii) $2x\,e^{x^2}\sin^{2}x + 2e^{x^2}\sin x\cos x$
   (iv) $2e^{2x}/x^2 - 2e^{2x}/x^3$

2. $-\cdot 0396$.

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